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GREEN'S INTEGRALS AND THEIR APPLICATIONS TO ELLIPTIC SYSTEMS

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CONTENTS

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INTRODUCTION

The theory of elliptic complexes of linear partial differential operators is closely interwoven with complex analysis. In particular, the Dolbeault complex is at the same time an important example of an elliptic complex and a tool for investigating the properties of more general ones. Although several results of complex analysis do not extend to arbitrary elliptic complexes, it is worthwhile to pursue those ideas and methods that have suitable extensions to the general theory.

This thesis is mainly concerned with integral representations. They were introduced and successfully used to study several problems in complex analysis (see [AYu], [He]), like obtaining homotopy formulae for $\overline{\partial}$-complex, sharp estimates for the $\overline{\partial}$-Neumann-Spencer problem, results on the theory of CR-functions, approximation theorems and removability of singularities of holomorphic functions and had also applications to complex integral geometry and to other subjects.

The formula of Martinelli-Bochner (see, for example, [AYu], [Ky]) provides one of the simplest integral representations for holomorphic functions defined in a bounded domain D of the n-dimensional complex space $\mathbb{C}^n$. The values of a holomorphic function in D are expressed by integrating its values on ∂D against a kernel which is relatively simple and has a general expression, independent of the domain. This kernel coincides with the Cauchy kernel in the case of one complex variable, but is not holomorphic with respect to the "exterior" variables in $\mathbb{C}^n$: this fact can be taken as one of the reasons of the deep difference between complex analysis in one and several complex variables.

The Martinelli-Bochner formula was employed to study properties of the CR-functions and the $\overline{\partial}$-Neumann problem for functions (see [Ky]), the Cauchy problem for the Cauchy-Riemann system (see [AKy], [ShT4]) and the solvability of inhomogeneous Cauchy-Rie- mann system (see [Rom2]).

In the theory of partial differential equations the method of integral representations is mainly related to the construction and use of parametrices (see [T5]). In this research I am concerned with elliptic systems , both determined and overdetermined. They admit, at least locally, left fundamental solutions. Green's integrals associated to them (see, for instance, [T5]) are natural analogues of the Martinelli-Bochner integral of complex analysis.

In this dissertation I apply Green's integrals to the Cauchy problem for elliptic systems and to the question of the validity of the Poincaré Lemma for elliptic differential complexes.

Let me describe more precisely the contents of the thesis.

Let X be an open set of the Eucledian space $\mathbb{R}^n$ and $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ be (trivial) $\mathbb{C}$-vector bundles over X. If $\mathfrak{C}$ is a class of distributions, the space $\mathfrak{C}(E_{|\sigma})$ of sections of E over an open subset σ of X is naturally identified with the space $[\mathfrak{C}(\sigma)]^k$ of k-columns of objects from $\mathfrak{C}(\sigma)$, and similarly for F. For every element P of the space $do_p(E \to F)$ of the linear differential operators with ($C^\infty-$) smooth coefficients and of order $\leq p$ between the vector bundles E and F, we have an expression $P(x,D) = \sum_{|\alpha|\leq p} P_\alpha(x) D^\alpha$, where $P_\alpha(x)$ are $(l \times k)$-matrices of (infinitely) differentiable functions on X. The expression $\sigma(P)(x,\zeta) = \sum_{|\alpha|=p} P_\alpha(x)\zeta^\alpha$ (for $x \in X$, $\zeta \in \mathbb{C}^n$) is called the principal symbol of $P \in do_p(E \to F)$. We say that P has an injective symbol if the matrix $\sigma(P)(x,\zeta)$ has rank k for every $(x,\zeta) \in X \times \mathbb{R}^n\backslash\{0\}$.

An important class of operators with injective symbols is the class of determined elliptic differential operators of order p (corresponding to the case $l = k$). The classic examples of overdetermined systems with injective symbol are the gradient operator in $\mathbb{R}^n$ and the Cauchy-Riemann system in $\mathbb{C}^n$, if $n > 1$.

As in the classic examples, under not too restrictive assumptions on P, it is possible to include it into some elliptic complex of linear partial differential operators on X, say, $\{E^i, P^i\}$ where $E^i = X \times C^{k_i}$ are (trivial) $\mathbb{C}$-vector bundles over X with $k_i \neq 0$ only for finitely many indexes i, $P^i \in do_{p_i}(E^i \to E^{i+1})$ and $P^0 = P$ (see Samborskii [Sa]). I shall often use this identification, assuming therefore that P satisfies suitable addition assumptions in [Sa].

If the differential operator P has injective symbol then P is hypoelliptic: for every distribution $u \in \mathcal{D}'(E)$ the singular supports of u and Pu coincide. In particular, all solutions of the system $Pu = 0$ on an open subset σ of X, belonging to $\mathcal{D}'(E_{|\sigma})$ agree in the sense of distributions with sections from $C^\infty(E_{|\sigma})$.

The Cauchy problem for solutions of the system $Pu = 0$ in a relatively compact domain D in X with a sufficiently smooth boundary, and data on a set S of positive ($(n-1)$-dimensional) measure on the boundary, can be roughly formulated as follows.

Problem 1. *Let u_α $(|\alpha| \leq p-1)$ be given sections of E over S. It is required to find a solution u of the equation $Pu = 0$ in D whose derivatives $D^\alpha u$ up to order $(p-1)$ have, in a suitable sense, boundary values $(D^\alpha u)_{|S}$ on S satisfying $(D^\alpha u)_{|S} = u_\alpha$ $(|\alpha| \leq p-1)$.*

Since the time of Hadamard, this problem has been known as a classic example of an ill-posed problem (see Hadamard [Hd], p.39). However, it naturally arises in the applications (see Hadamard [Hd], p.38). For example, the Cauchy problem for the Laplace operator naturally arises in problems of the interpretation of electrical prospecting data.

The Cauchy problem for the Laplace operator, in various formulations, has been studied by Mergeljan [Me], Lavrent'ev [Lv1],[Lv3], Ivanov [Iv], Newman [Ne], Koroljuk [Kor], Maz'ya and Havin [MzHa], Jarmuhamedov [Ja], Shlapunov [Sh1], [Sh5], and others. For holomorphic functions of one variable the Cauchy problem was considered in the papers of Carleman [Ca], Zin [Zin], Fok and Kuny [FKun], Patil [Pa], Krein and Nudelman [KrNu], Steiner [Str], and of other mathematicians. The Cauchy problem for the overdetermined Cauchy-Riemann system was studied by Tarkhanov [T2], Znamenskaya [Zn], Aizenberg and Kytmanov [AKy], Karepov and Tarkhanov [KT1], Karepov [K], Shlapunov and Tarkhanov [ShT4], and others. The Cauchy problem for the Lamé system (related to the theory of linear elasticity) was studied by Mahmudov [Ma] and Shlapunov [Sh4], [Sh5]. The Cauchy problem for general systems of linear partial differential equations with injective symbols has been investigated by Tarkhanov [T1]-[T4], Nacinovich [Na], and others.

The present research is an attempt to elucidate how the use of bases with double orthogonality (see Slepian and Pollak [SlPo], Landau and Pollak [LPo1], [LPo2], Slepian [Sl]) and of Green's integrals gives new insight into the Cauchy problem for general systems of linear partial differential equations with injective symbols. More precisely, in terms of the bases with double orthogonality, we obtained solvability conditions for the ill-posed Cauchy problem which are constructive, and simpler and more convenient than those previously known (see Tarkhanov [T2]). Essentialy these conditions consist of the convergence of a Fourier series of a potential (Green's integral), associated with the Cauchy data, with respect to a basis with the double orthogonality property. Moreover, a constructive formula for the regularization (approximate solution) of the Cauchy problem for general systems of differential equations with injective symbols has been devised. Earlier it was proved that such a regularization (Carleman's type formula) existed (see Tarkhanov [T1])). But the possibility of a constructive approach was devised only for the Cauchy-Riemann system, or, more generally, for systems factorizing the Laplace operator (see Aizenberg [A], Jarmuhamedov [Ja], Mahmudov [Ma], and others).

The results on the Cauchy problem are described in Chapter 2; they are essentially due to Shlapunov and Tarkhanov (see [Sh1], [Sh4], [Sh5], [ShT2], [ShT3], [ShT4]).

In order to study the Cauchy problem we need to obtain informations on the boundary behaviour of solutions of elliptic systems. These are provided by theorems on the jump of Green's type integrals. They imply that the regularity of a solution u of Problem 1 near S is completely determined by the smoothness of the Cauchy data u_α ($|\alpha| \leq p-1$). In particular, if $u_\alpha \in C^{p-1-|\alpha|}(E_{|\overset{\circ}{S}})$ (where $\overset{\circ}{S}$ is the interior of S in ∂D) then $u \in C^{p-1}_{loc}(S \cup D)$ (see §1.3 below). The regularity of u near the points of $\partial D \backslash S$ is determined by that class of functions (sections) in which we seek the solution of the Cauchy problem. These topics are discussed in Chapter 1, where also the background material and the relevant definitions are collected.

The last part of this thesis is centered on the question of the validity of the Poincaré lemma, i.e. local acyclicity, for elliptic complexes of linear partial differential operators with smooth coefficients. This is a long standing problem of the theory of overdetermined systems (see [T5], [AnNa]). For (not necessarily elliptic) complexes with constant coefficients the Poincaré lemma is always valid (see [Pal] and [Ml1], [Ml2]); it also holds for elliptic complexes with real analytic coefficients (see [AnNa]).

Although we are still not able to settle the question whether the Poincaré lemma is valid, we succeed in Chapter 3 in proving a representation formula for solutions of the system $Pu = f$ for an operator P with injective symbol whenever they exist.

This representation involves the sum of a series whose terms are iterations of integro- differential operators (in particular, Green's integrals), while solvability of the equation $Pu = f$ is equivalent to the convergence of the series together with the orthogonality to a harmonic space (the last one is a trivial necessary condition).

For the Dolbeault complex, these integro-differential operators are related to the Mar- tinelli-Bochner integral. In this case, results similar to ours were obtained by Romanov [Rom2]. In the general situation these results are obtained by Nacinovich and Shlapunov (see [NaSh]). In fact this approach is more fit to study the global solvability of the system $Pu = f$ (cf. [Sh3], [Sh6]). æ

CHAPTER I

GREEN'S INTEGRALS AND BOUNDARY BEHAVIOUR OF SOLUTIONS OF ELLIPTIC SYSTEMS

§1.0. Introduction

In this chapter we use Green's integrals to study the boundary behaviour of solutions of elliptic systems. Theorems on the jump of an integral of Green's type with density in various classes of distributions are extremely useful for this purpose. In fact, theorems of this kind are to some extent analogues of the Sokhotsky formulae for the Cauchy integral.

§1.1 consists of preliminary information about Green's integrals (such as important notions, definitions and simple properties).

As an example of dealing with Green's integrals, in §1.2 we investigate the jump behaviour of the Martinelli-Bochner integral. In particular, we discuss Privalov's Principal Lemma for the Martinelli-Bochner integral (see [Ky]) and a "delicate" theorem on the jump behaviour of this integral at special generalized Lebesgue points of a summable density (cf. [Sh4]).

§1.3 is devoted to the investigation of the weak boundary values of solutions of an elliptic system which have finite order of growth near the boundary. In particular, we prove the theorem on the (weak) jump of Green's integrals for general elliptic systems (see also [ShT2]). This theorem is one of the principal tools of the present approach.

The application of bases with double orthogonality to the Cauchy problem (see Chapter 2 below) dictates to which class a solution belongs. This turns out to one of the Sobolev spaces $W^{m,2}$. In §1.4 we investigate weak boundary values of the solutions in the Sobolev class $W^{m,q}(E_{|D})$ (cf. [ShT2]). Essentially these results are due to Rojtberg [Roj]. Our slight modifications concern overdetermined systems.

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§1.1. Green's operators and Green's integrals

Let $X \subset \mathbb{R}^n$ be an open set, $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ be (trivial) vector bundles over X. Sections of E and F of a class $\mathfrak{C}$ on an open set $\sigma \subset X$ can be interpreted as columns of complex valued functions from $\mathfrak{C}(\sigma)$, that is, $\mathfrak{C}(E_{|\sigma}) \cong [\mathfrak{C}(\sigma)]^k$, and similarly for F. Throughout the thesis we will mostly use the letters u, v for sections of E, and the letters f, g for sections of F.

We denote by $C^m_{loc}(E_{|\sigma})$ $(m \geq 0)$ the vector space of m times continuously differentiable functions on σ, endowed with the usual topology (of uniform convergence on compact subsets of σ together with all the derivatives up to order m). We denote also by $C^\infty_{loc}(E_{|\sigma})$ $(= \mathcal{E}(E_{|\sigma})\ )$ the space of infinitely differentiable functions on σ, endowed with the usual

Frechet-Swartz topology and by $C^\infty_\circ(E_{|\sigma})$ $(= \mathcal{D}(E_{|\sigma}))$ the space of infinitely differentiable functions with compact supports on σ, topologized in the usual way. The spaces $C^m_{loc}(E_{|\sigma})$ and $\mathcal{E}(E_{|\sigma})$ are well known to be Frechet space. Sometimes we will write simple $C^m(E_{|\sigma})$ for $C^m_{loc}(E_{|\sigma})$.

We will say that $u \in C^m(E_{|\Omega})$, with a (not necessary open) set $\Omega \subset \mathbb{R}^n$, if $u \in C^m_{loc}(E_{|\overset{\circ}{\Omega}})$ continuously extends together with its derivatives up to order m to Ω. Then $C^m(E_{|\Omega})$ is a Frechet space too (end even Banach space, if $\Omega \subset \mathbb{R}^n$ is a compact and $m < \infty$).

As usual, $\mathcal{D}'(E_{|\sigma})$ is the space of distributions and $\mathcal{E}'(E_{|\sigma})$ is the space of distributions with compact supports on σ.

Further, let E^* be the dual bundle of E, and let $(.,.)_x$ be a Hermitian metric in the fibers of E. Then $*_E : E \to E^*$ is defined by $< *_E v, u >_x = (u,v)_x$ (where u, v are sections of E and $< w, u >_x = \sum_j^k w_j(x) u_j(x)$ is the natural pairing $E^* \otimes E \to \mathbb{C}$). Let Λ^r be the bundle of complex valued exterior forms of degree r $(r = 0, 1, \dots)$ over X, and dx the usual volume form on X.

Let $L^q(E_{|D})$ (with $1 \le q < \infty$) be the Banach space of all measurable functions defined on D, for which

$$\|u\|_{L^q(E_{|D})} = \left(\int_D (u,u)_x^{q/2} dx \right)^{1/q} < \infty.$$

We also will denote by $W^{m,q}(E_{|D})$ the Sobolev space of distribution sections of E over D having weak derivatives in the Lebesgue space $L^q(E_{|D})$ up to order m. The space $W^{m,q}(E_{|D})$ is the Banach space with the norm

$$\|u\|_{W^{m,q}(E_{|D})} = \left(\sum_{|\alpha| \le m} \int_D (D^\alpha u, D^\alpha u)_x^{q/2} dx \right)^{1/q}.$$

As usual, the sign $W^{m,q}_{loc}(E_{|D})$ we will use for the Sobolev space of functions belonging to $W^{m,q}(E_{|K})$ for any compact set $K \Subset D$ $(L^q_{loc}(E_{|D}) = W^{0,q}_{loc}(E_{|D}))$.

Let $do_p(E \to F)$ be the vector space of smooth linear partial differential operators of order $\le p$ between the vector bundles E and F. Then $P \in do_p(E \to F)$ is an $(l \times k)$ matrix of scalar linear partial differential operators, i.e. we have

$$P(x, D) = \sum_{|\alpha| \le p} P_\alpha(x) D^\alpha$$

where $P_\alpha(x)$ are $(l \times k)$-matrices of smooth functions on X. We will denote by $\sigma(P)$ its principal symbol

$$\sigma(P)(x, \zeta) = \sum_{\alpha = p} P_\alpha(x) \zeta^\alpha \ (x \in X, \ \zeta \in \mathbb{R}^n).$$

An open set is the natural domain of the system $Pf = 0$. However some problems require the consideration of solutions on sets $\sigma \subset X$ which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the so-called local solutions of the system $Pu = 0$ on σ, that is, solutions of this system in a neighbourhood of σ. The space of local solutions of the system $Pu = 0$ on σ will be denoted by $S_P(\sigma)$ and let $S_P^{m,q}(D) = W^{m,q}(E_{|D}) \cap S_P(D)$ be the closed linear subspace of $W^{m,q}(E_{|D})$ of weak solutions of the equation $Pu = 0$ in D.

We will denote by ${}^tP \in do_p(F^* \to E^*)$ (or by P' where it is more convenient) the transposed operator

$$P'(x, D) = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha ({}^tP_\alpha(x) \times \cdot),$$

and by $P^* = (*_E^{-1} P' *_F) \in do_p(F \to E)$ the (formal) adjoint operator of $P \in do_p(E \to F)$. In the standard case, $(u, v)_x = \sum_{j=1}^k u_j(x)\overline{v}_j(x)$, $(f, g)_x = \sum_{j=1}^l f_j(x)\overline{g}_j(x)$, we have

$$P^*(x, D) = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha (P^*_\alpha(x) \times \cdot),$$

where $P^*_\alpha(x)$ is the conjugate matrix of $P_\alpha(x)$.

Definition 1.1.1. A differential bilinear operator $G_P(.,.) \in do_{p-1}((F^*, E) \to \Lambda^{n-1})$ is said to be Green's operator for $P \in do_p(E \to F)$ if the following formula holds:

$$dG_P(g, v) = < g, Pv >_x dx - < {}^tPg, v >_x dx \ (g \in C^\infty(F^*), v \in C^\infty(E)).$$

For the proof of the following properties of Green's operators we refer readers to the book of Tarkhanov [T5] (pp. 82-83).

Proposition 1.1.2. *Green's operator for a differential operator $P \in do_p(E \to F)$ always exist. Moreover, if G^1 and G^2 are two Green's operators for P then there exists a bidifferential operator $T \in do_{p-2}((F^*, E) \to \Lambda^{n-2})$ that $G^2 - G^1 = dT$.*

Proposition 1.1.3. *If $P \in do_p(E \to F)$ and $Q \in do_q(F \to G)$ with a trivial vector bundle $G = X \times \mathbb{C}^m$ then*

$$G_{{}^tP}(v, g) = \overline{G_{P^*}(*v, *^{-1}g)} = -G_P(g, v),$$
$$G_{QP}(g, v) = G_Q(g, Pv) + G_P({}^tQg, v).$$

For instance, Green's operator G_P can be written in the form

$$G_P(g, v) = \sum_{|\beta+\gamma+1_j| \leq p}' (-1)^{|\beta|} D^\beta (g P_{\beta+\gamma+1_j}) D^\gamma v (*dx_j) \tag{1.1.1}$$

where $\sum'$ indicate that an order has been selected with respect to the multi-indexes $\beta, \gamma, 1_j$, and $*$ is the Hodge operator defined for differential forms (see [T5], p.82). In particular,

there exists the only one Green's operator G_P for a first order differential operator P; this operator is given by the following formula:

$$G_P(g, v) = g \left[\sigma(P) \left(x, \frac{*\partial x}{\sqrt{-1}} \right) \right] v,$$

where $*\partial = (*dx_1, \ldots, *dx_n)$.

For the purposes of this research it is more convenient to write Green's operators in other form. However for this we need the so-called Dirichlet systems of boundary operators.

Let D be a relatively compact domain in X with smooth boundary, let U be a neighbourhood of ∂D in X, and $F_j = U \times \mathbb{C}^k$ $(0 \leq j \leq r < \infty)$ be (trivial) vector bundles over the neighbourhood U.

DEFINITION 1.1.4. A system $\{B_j\}_{j=0}^r$ of differential operators $B_j \in do_{b_j}\ E_{|U} \to F_j)$ is said to be a Dirichlet system of order r on ∂D if 1) $0 \leq b_j \leq r$; 2) $b_j \neq b_i$ for $j \neq i$; 3) $rank_{\mathbb{C}} \sigma(B_j)(y, d\rho) = k$ $(0 \leq j \leq r), y \in U$, where $\sigma(B_j)$ is the principal symbol of the operator B_j, and ρ belongs to the class of functions defining the domain D $(D = \{x \in X : \rho(x) < 0,\ |d\rho| \neq 0 \ in\ U\})$.

The following proposition shows how important for various boundary value problems the Dirichlet systems are. One can find similar statement, for example, in the books [Bz] and [T4] (Lemma 28.2).

PROPOSITION 1.1.5. *Let $\partial D \in C^s$, $s \geq r$, and $\{B_j\}_{j=0}^r$ is a Dirichlet system of order r in U. Then for any system of sections $u_j \in C^{s-b_j}(F_{j|\partial D})$, $0 \leq j \leq r$, there is a section $u \in C^s(E_{|\overline{D}})$ such that $B_j u_{|\partial D} = u_j$ for $0 \leq j \leq r$.*

The following lemma was proved in [T4] (p.280, Lemma 28.3).

LEMMA 1.1.6. *Suppose that the boundary ∂D of D is non characteristic for $P \in do_p(E \to F)$ $(l \geq k)$. Then, given Dirichlet system $\{B_j\}_{j=0}^{p-1}$, one can find a neighbourhood U of ∂D, and Green's operator G_P such that*

$$G_P(g, v) = \sum_{j=0}^{p-1} < C_j g, B_j v >_x ds + \frac{d\rho}{|d\rho|} \wedge G_\nu(g, v) \ (g \in C^\infty(F^*_{|U}), v \in C^\infty(E_{|U}))$$

where $\{C_j\}_{j=0}^{p-1}$ is a Dirichlet system of order $(p-1)$ on ∂D such that $C_j \in do_{p-b_j-1}(F^*_{|U} \to F^*_j)$ $(0 \leq j \leq p-1)$, and $G_\nu \in do_{p-1}((F^*, E)_{|U} \to \Lambda^{n-2})$.

Using Green's operators one obtains integral representations for solutions of the system $Pu = 0$.

A matrix $\mathcal{L}(x, y)$ is said to be a left fundamental solution of the operator $P \in do_p(E \to F)$ on X if

$$\int_X < \mathcal{L}(x, y), P(y)v(y) >_y dy = v(x) \ for \ every \ v \in C^\infty_\circ(E);$$

and a matrix $\mathcal{R}(x, y)$ is said to be a right fundamental solution of the operator $P \in do_p(E \to F)$ on X if

$$P(x) \int_X < \mathcal{R}(x, y), g(y) >_y dy = g(x) \ for \ every \ g \in C^\infty_\circ(F).$$

We say that the linear partial differential operator $P \in do_p(E \to F)$ is elliptic if its principal symbol

$$\sigma(P)(x, \zeta) : \mathbb{C}^k \to \mathbb{C}^l$$

is injective for every $x \in X$ and $\zeta \in \mathbb{R}^n \backslash \{0\}$. In particular $l \geq k$; we say that P is determined elliptic if $l = k$ and overdetermined elliptic if $l > k$. Every determined elliptic operator with smooth coefficients has locally a bilateral (i.e. left and right) fundamental solution, and hence every overdetermined elliptic operator with smooth coefficients has locally a left fundamental solution. If the coefficients of the operator P are real analytic, there exist global fundamental solutions of the operator P on X (cf., for example, [T5], §8). In fact, for the existence of left (right) fundamental solutions of the operator P, the so-called Uniqueness Condition $(U)_S$ for the Cauchy problem in small on X for the operator P (P') is important (see [T4], Corollary 27.8):

(U)$_S$ *if for a domain $O \subset X$ we have $Pu = 0$ in O, and $u = 0$ on a non-empty open subset of O then $u \equiv 0$ in O.*

From now on we will assume that the operator P is elliptic.

THEOREM 1.1.7 (GREEN'S FORMULA). *Let $\mathcal{L}$ is a (left) fundamental solution of the operator P on X. For every $u \in W^{p,2}(E_{|D})$ the following formula holds:*

$$
-\int_{\partial D} G_P(\mathcal{L}(x,y), u(y)) + \int_D < \mathcal{L}(x,y), Pu(y) >_y dy = \begin{cases} u(x), x \in D, \\ 0, x \in X \backslash \overline{D}. \end{cases} \tag{1.1.2}
$$

PROOF. If $u \in C^p(E_{|\overline{D}})$ (that is, u is p times continuously differentiable in a neighbourhoud of $\overline{D}$) then (1.2) follows from the Stokes' formula and Definition 1.1. Since the boundary of D smooth, there exists a sequence of functions $\{u_N\}_{N=1}^{\infty} \in C^p(E_{|\overline{D}})$ approximating u in $W^{p,2}(E_{|D})$. Then, for every number $N \in \mathbb{N}$,

$$
-\int_{\partial D} G_P(\mathcal{L}(x,y), u_N(y)) + \int_D < \mathcal{L}(x,y), Pu_N(y) >_y dy = \begin{cases} u_N(x), x \in D, \\ 0, x \in X \backslash \overline{D}. \end{cases} \tag{1.1.3}
$$

Using the boundedness theorem for pseudo-differential operators (see [ReSz], 1.2.3.5) we conclude that the second integral in the left hand side of (1.1.2) is a bounded linear operator from $W^{p,2}(E_{|D})$ to $W^{p,2}(E_{|D})$.

Thus, to obtain (1.1.2) it suffices to pass to the limit in (1.1.3) for $N \to \infty$. □

The integrals of the type:

$$
-\int_{\partial D} G_P(\mathcal{L}(x,y), u(y))
$$

we will call Green's integrals associated to the operator P and denote by $\mathcal{G}u$.

Similar (to Theorem 1.1.7)) results could be obtained for various classes of functions. For example, see Corollary 10.1 in the book [T5], or Theorem 1.3.2 below.

Remark 1.1.8. The boundary integral in the left hand side of (1.1.2) does not depend on the choice of Green's operator G_P (see Proposition 1.1.2).

COROLLARY 1.1.9. *Let $\mathcal{L}$ be a bilateral fundamental solution of the operator P on X. Then the boundary integral in (1.2) is a (bounded) projection from $W^{m,2}(E_{|D})$ onto $S_P^{m,2}(D)$; and for every $f \in W^{m-p,2}(F_{|D})$ $(m \geq p)$ the integral $\int_D < \mathcal{L}(x,y), f(y) >_y dy$ is a $W^{m,2}(E_{|D})$-solution of the equation $Pu = f$ in D.*

PROOF. Since the derivatives $D^\alpha u$ $(|\alpha| \leq p-1)$ have natural boundary values $D^\alpha u_{|\partial D} \in W^{m-|\alpha|-1/2,2}(E_{|\partial D})$ (see [EgSb], p.120), it is easy to see from Proposition 1.1.2 that the boundary integral in (1.2) does not depend on the choice of Green's operator G_P. Therefore, choosing as G_P Green's operator provided by Lemma 1.6.6, and using boundedness theorem for potential (co-boundary) operators on a manifold with boundary ([ReSz], 2.3.2.5) one can conclude that the boundary integral in (1.1.2) defines a bounded linear operator from $W^{m,2}(E_{|D})$ to $W^{m,2}(E_{|D})$. Hence the statement follows from the properties of bilateral fundamental solutions of elliptic differential operators. □

Remark 1.1.10. All the discussion above we can repeat with small technical changes under weaker smoothness assumption on ∂D. Namely, according to the usual understanding, differential operators on X must have (infinitely) differentiable coefficients, however the smoothness of the coefficients of the differential operators $\{C_j\}$ and G_ν (in Lemma 1.1.6) is finite if the smoothness of boundary is finite (see [T4], p.280). One may check what smoothness requirements for the coefficients of $\{C_j\}$ are satisfied as a consequence of the supposed smoothness of the boundary of D (and coefficients of the initial expressions $\{B_j\}$). Certainly, these difficulties are removed if $\partial D \in C^\infty$. For our purposes it is sufficient that the coefficients of every differential operator B_j belong to the class $C^{p-1-b_j}_{loc}$, and the coefficients of each differential operator C_j belong to the class C^{b_j} in the neighbourhood U.

Without loss of a generality we assume that $b_j = j$. For example, we can set $B_j = I_k \frac{\partial^j}{\partial n^j}$, where $\frac{\partial^j}{\partial n^j}$ is the j-th normal derivative with respect to ∂D and I_k is the unit $(k \times k)$-matrix.

If the operator P is overdetermined, it may happen that there are no right (in particular bilateral) fundamental solutions.

Example 1.1.11. If P is the Laplace operator Δ_n in $\mathbb{R}^n$, then there is a (bilateral) fundamental solution of P, for example, the standard fundamental solution φ_n of the convolution type. In this case (1.1.2) is Green's formula for harmonic functions and the boundary integral in (1.1.2) is the well-known Green's integral.

Example 1.1.12. If P is the Cauchy-Riemann system $\frac{d}{d\bar{z}}$ in $\mathbb{C}^1$ ($\cong \mathbb{R}^2$), then there is a (bilateral) fundamental solution $\mathcal{L}(\zeta, z) = \frac{1}{\pi(\zeta - z)}$ of P. In this case (1.1.2) is the Cauchy-Green formula (see [He]) and the boundary integral in (1.1.2) is the well-known Cauchy integral.

Example 1.1.13. If P is the Cauchy-Riemann system $\overline{\partial}$ in $\mathbb{C}^n$ ($\cong \mathbb{R}^{2n}$), $n > 1$, then there are no right fundamental solutions of P (due to the theorem on removability of compact singularities of holomorphic functions in $\mathbb{C}^n$ of dimension $n > 1$). As a left fundamental solution of the Cauchy-Riemann system we can take $\mathcal{L}(\zeta, z) = {}^tP^*(\zeta)\varphi_{2n}(\zeta, z)$ where φ_{2n} is the standart fundamental solution of the Laplace operator in $\mathbb{R}^{2n}$ and $\zeta, z \in \mathbb{C}^n$. In this case (1.1.2) is the Martinelli-Bochner formula (see [AYu]) and the boundary integral in (1.1.2) is the Martinelli-Bochner integral. It is known that the Martinelli-Bochner integral is only harmonic (but, in general, not holomorphic) everywhere outside of ∂D. Hence it is not a projection from $W^{m,2}(E_{|D})$ onto $S^{m,2}_{\overline{\partial}}(D)$. Moreover, the integral $\int_D < \mathcal{L}(x, y), f(y) >_y dy$ is not a solution of the equation $\overline{\partial} u = f$ in the domain D.

Romanov [Rom2] proved that, if D is a bounded domain in $\mathbb{C}^n$, the limit $\lim_{\nu\to\infty} M^\nu$ of iterations of the Martinelli-Bochner integral M in the Sobolev space $W^{1,2}(D)$ exists; and that this limit is a projection from $W^{1,2}(D)$ onto the space of holomorphic $W^{1,2}(D)$ - functions (i.e. onto $S^{1,2}_{\overline{\partial}}(D)$). Using the iterations he also obtained a multi-dimensional analogue of the Cauchy-Green formula in the plane, and, as a corollary, an explicit formula for solutions of the equation $\overline{\partial} u = f$ in pseudo-convex domains in $\mathbb{C}^n$.

Fortunately, the phenomenon of convergence of iterations of the boundary integral in (1.1.2) is more general than a particular property of the Martinelli-Bochner integral. For example, in [Sh3] the theorem on iterations was proved for special Green's integrals of matrix factorizations of the Laplace operator in $\mathbb{R}^n$ and in [NaSh] for special Green's integrals associated to operators with injective symbols (see also Chapter 3 below). æ

§1.2. On the jump behaviour of the Martinelli-Bochner integral

As we have seen in Example 1.1.13, the Martinelli-Bochner integral is nothing but a Green integral associated with the Cauchy-Riemann system. In this section we will illustrate on the example of the Martinelli-Bochner integral how to deal with Green's integrals for general elliptic systems.

The classical notions of maximum function and Lebesgue point for summable functions are very important in modern theory of functions (see, for example, [St]). Information on the boundary behaviour of the Martinelli-Bochner integral in Lebesgue points of a summable density one can extract from Privalov's Principal Lemma (see 1.2.1 below).

Also in this section we consider "delicate" theorems on the jump of the Martinelli-Bochner integral at generalized Lebesgue points of a summable function.

In 1.2.2 we give some definitions and lemmata concerned with one of possible generalization of the classical notions of maximum function and Lebesgue point of a locally summable function. It seems that these results are known and rather trivial. However we could not find an appropriate reference.

In 1.2.3 we investigate some properties of the Poisson integral of a summable function at the generalized Lebesgue points. For the usual Lebesgue points one can find some of the properties in [St], some results, for example Theorem 1.2.3.2, were mentioned (without a reference) in a paper of Rudin [Ru].

We use the result of 1.2.2 and 1.2.3 to study in 1.2.4 the behaviour of Martinelli-Bochner integral with a density while the exterior variables of the integral crosses a smooth integration hypersurface. In particular, for generalized Lebesgue points we prove a theorem on the jump of the Martinelli-Bochner integral with bounded density. It is an analogue of the theorem for the usual Lebesgue points of summable functions (see 1.2.1 below and [Ky]). Also we study the jump behaviour of the Martinelli-Bochner integral with a continuous density given on a measurable subset S of a smooth closed hypersurface H (cf. [Sh2]).

1.2.1. The Privalov's Principal Lemma.

Let D be a domain in $\mathbb{C}^n$ $(n \geq 1)$ and H be a piece-wise smooth closed hypersurface dividing D onto two domains D^+ and D^-. We choose on H the orientation of ∂D^-. As usual, by "piece-wise smooth" we mean that H is the union of a finite number of pieces of smooth hypersurfaces H_k, which intersect transversally. To each point $z^0 \in H$ we associate two nondegenerate tangential cones $T(D^\pm, z^0)$. They are supplementary whose boundaries are contained in the tangent hyperplanes to the H_k at z^0.

We denote by $B(z^0, r)$ the ball of radius r in $\mathbb{R}^{2n}$ $(\cong \mathbb{C}^n)$ centered at the point z^0, and by $m(B(z^0, r))$ its Lebsgue measure. Sometimes we will simply write $B(r)$ for the ball with centre at zero.

Also we denote by $\alpha(D^\pm, z^0)$ the ratio of the measure of the solid angle of the tangential cone $T(D^\pm, z^0)$ on the $(2n-1)$-dimensional measure of the unit $(2n-1)$-

sphere in $\mathbb{C}^n$:

$$\alpha(D^{\pm}, z^0) = \lim_{r\to 0}\left(\int_{\partial B(z^0,r)\cap D^{\pm}} ds\right) \Big/ \left(\int_{\partial B(z^0,r)} ds\right) \quad (z^0 \in H),$$

and by $K^{\pm}(z^0)$ cones with tip at z^0 belonging respectively to $T(D^{\pm}, z^0)$ and only intersecting at the tip z^0.

Let us rehearse some known definitions and facts to be compared with their generalizations in 1.2.2 below.

DEFINITION 1.2.1.1. A point $x \in \mathbb{R}^n$ is said to be a Lebesgue point of a locally summable function $f \in L^1_{loc}(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0.$$

DEFINITION 1.2.1.2. Let $f \in L^1(\mathbb{R}^n)$. Then we will say that the function

$$mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dy$$

is the maximum function of the function f.

Let now

$$\mathfrak{U}(z,\zeta) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{\zeta}_k - \overline{z}_k}{|\zeta - z|^{2n}} d\overline{\zeta} \wedge d\zeta$$

be the Martinelli-Bocner kernel in $\mathbb{C}^n$, and

$$Mf(z) = \int_H \mathfrak{U}(z,\zeta) f(\zeta)$$

be the Martinelli-Bochner integral with a summable density $f \in L^1(H)$. In general, at a point $z \in H$ this integral with density of such a class does not exist as singular integral, because the integrand has a point-wise singularity of order which is equal to the dimension of H. Moreover, even if f is continuous, this integral may not exist as the limit

$$V.P.\ Mf(z^0) = \lim_{\varepsilon\to 0} \int_{H\setminus B(z^0,\varepsilon)} \mathfrak{U}(z^0,\zeta) f(\zeta).$$

We will study in this section the behaviour of the integral M near the hypersurface H.

LEMMA 1.2.1.3. *For every point* $z^0 \in H$

$$V.P. \int_{\partial D^-} \mathfrak{U}(z^0,\zeta) = \alpha(D^-, z^0).$$

PROOF. See, for example, book [Ky]. □

The following result was obtained by Kytmanov [Ky]. We omit its proof because in 1.2.4 we will use similar arguments in order to prove a more "delicate" theorem on the jump behaviour of Martinelli-Bochner integral.

THEOREM 1.2.1.4 (PRIVALOV'S PRINCIPAL LEMMA). *Let* $f \in L^1(H)$ *and* $z^0 \in H$ *be a Lebesgue point of the function* f. *Then*
(1.2.1.1)

$$Mf(z) = \mp\alpha(D^{\mp}, z^0) f(z^0) + \int_{H\setminus B(z^0,|z-z^0|)} \mathfrak{U}(z,\zeta) f(\zeta) + r(z,z^0)\ (z \in D),$$

where $r(z,z^0) \to 0$ *if* $z \to z^0$ *in the cone* $K^{\pm}(z^0)$. *Moreover, for every compact set* $K \subset H$ *there is a constant* $C > 0$ *such that* $|r(z,z^0)| \le C\, mf(z^0)$ *for all* $z^0 \in K$ *and* $z \in K^{\pm}(z^0)$.

The following formula for the jump of the Martinelli-Bochner integral at Lebesque points of the density f follows immediately from Theorem 1.2.1.4.

COROLLARY 1.2.1.5. *Let* $f \in L^1(H)$ *and* $z^0 \in H$ *be a Lebesgue point of the function* f. *Then*

$$\lim_{z^+,z^-\to z^0} (Mf(z^-) - Mf(z^+)) = f(z^0),$$

where $z^{\pm} \in K^{\pm}(z^0)$ *and the limit is uniform on compact subsets of* H.

We will discuss in 1.2.4 formulae of this type for the Martinelli-Bochner integral at generalized Lebesque point of the density f.

The jump behavior of Green's integrals with smooth densities and distribution densities will be discussed in §1.3. For the jump behavior of Green's integrals with summable densities we refer the readers to the book [T4] (see Lemma 28.11 and Remark 28.12).

Let us show how to use the Privalov's Principal Lemma in order to obtain more detail information about boundary behaviour of Martinelli-Bochner integral.

We assume that D^- is a bounded domain in $\mathbb{C}^n$ with boundary $\partial D^- = H$ in C^1. Then Theorem 1.2.1.4 implies that, for all the Lebesgue points $z^0 \in \partial D^-$ of the function f for which there exists the singular integral (see [St], p.52)

$$V.P. \int_{\partial D^-} \mathfrak{U}(z^0,\zeta) f(\zeta)$$

(i.e. almost every where on ∂D^-) the integral $Mf(z)$ $(z \in D^\pm)$ has non tangential limit values $M_b f^\pm$ on H:

$$M_b f^\pm(z^0) = \mp f(z^0)/2 + V.P. \int_{\partial D^-} \mathfrak{U}(z^0, \zeta) f(\zeta) \ (z^0 \in H). \tag{1.2.1.2}$$

If $f \in L^q(H)$ $(1 < q < \infty)$ then, using boundedness theorem for singular integral operators in these spaces (see [St], p.52) and (1.2.1.2), we conclude that $M_b f^\pm \in L^q(H)$.

The following corollary clarify the character of the convergence of $Mf^\pm$ to its boundary limit values $M_b f^\pm$ on H (cf. [ShT4]).

COROLLARY 1.2.1.6. *Let $f \in L^q(H)$ $(1 < q < \infty)$. Then*

$$\lim_{\varepsilon \to 0} \left(\int_H |Mf(z \pm \varepsilon\nu(z)) - M_b f^\pm(z)|^q ds(z) \right)^{1/q} = 0. \tag{1.2.1.3}$$

PROOF. Let us prove the convergence of $Mf(z - \varepsilon\nu(z))$ (another proof is similar). Using formulae (1.2.1.1) and (1.2.1.2) for $z \in H$ and $\varepsilon > 0$ we obtain

$$Mf(z - \varepsilon\nu(z)) - M_b f^\pm(z) =$$

$$= \int_{H \setminus B(z^0, \varepsilon)} \mathfrak{U}(z, \zeta) f(\zeta) - V.P. \int_H \mathfrak{U}(z, \zeta) f(\zeta) + r(z, z - \varepsilon\nu(z)).$$

It is known in Theory of singular integral operators (see [St], p.52) that

$$\lim_{\varepsilon \to 0} \left(\int_{\partial D^-} \left| \int_{H \setminus B(z^0, \varepsilon)} \mathfrak{U}(z, \zeta) f(\zeta) - V.P. \int_H \mathfrak{U}(z, \zeta) f(\zeta) \right|^q ds(z) \right)^{1/q} = 0. \tag{1.2.1.4}$$

On the other hand, by Theorem 1.2.1.4, for the compact set $H = \partial D^-$ there is a constant $C > 0$ that

$$|r(z, z - \varepsilon\nu(z))| \le C m f(z)$$

for all $z \in H$ and sufficiently small $\varepsilon > 0$. Since the maximum operator is continuous as a map from $L^q(H)$ to $L^q(H)$ for $1 < q < \infty$, then using the Lebesgue theorem on the possibility to change the sign of the limit passage and the sign of integral we conclude that

$$\lim_{\varepsilon \to 0} \left(\int_{\partial D^-} |r(z, z - \varepsilon\nu(z))|^q \, ds(z) \right)^{1/q} = 0. \tag{1.2.1.5}$$

In order to obtain (1.2.1.3) from (1.2.1.4) and (1.2.1.5) it is sufficient to use the triangle inequality for the norm in $L^q(H)$, which was to be proved. □

In conclusion of this section we will formulate one more interesting (from the author's point of view) corollary. With this purpose for $1 \le q \le \infty$ we denote by $H^q(D^\pm)$ the Hardy classes of harmonic functions in the domain $D^\pm$ (see [PKuz]), i.e. the set of such harmonic functions $g(z)$ that

$$\limsup_{\varepsilon \to 0} \int_{\partial D^-} |g(z \pm \varepsilon\nu(z))|^q < \infty. \tag{1.2.1.6}$$

Corollary 1.2.1.7. *Let $\partial D^- \in C^2$ and $f \in L^q(\partial D^-)$ with $1 < q < \infty$. Then $Mf^\pm \in H^q(D^\pm)$.*

Proof. It is known that Mf is a harmonic function everywhere outside of the hypersurface $H = \partial D^-$. As for inequality (1.2.1.6) for the integral Mf, it follows from Corollary 1.2.1.6 because every convergent sequence is bounded. $\square$

1.2.2. Generalized Lebesgue points.

In this subsection we will try to generalize the notions of the Lebesque points of a density f in order to extend the set of boundary points where the jump formula for the Martinelli-Bochner integral still holds.

We will consider a locally summable function f in $\mathbb{R}^n$ given everywhere. Let us formulate at first some definitions.

Definition 1.2.2.1. A point $x \in \mathbb{R}^n$ is said to be a generalized Lebesgue point of a locally summable function f if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy = f(x).$$

Of course, a Lebesgue point of the function f (see Definition 1.2.1.1) is a generalized Lebesgue point, but the opposite statement is wrong.

It would be natural to use in the definition above not only the system of balls but also a suitable system of "contracting" sets. It is known that it is possible for the usual Lebesgue points. The generalized Lebesgue points are more "delicate" objects. For them it is possible too, but only for bounded functions and special type of set's systems.

Let $\mathfrak{F}$ be a family of measurable sets in $\mathbb{R}^n$. We will say that the family is regular if for every set $\sigma \subset \mathfrak{F}$ there is an (open) ball $B \supset \sigma$ with centre at the origin such that $m(\sigma) \geq c\, m(B)$ with a positive constant c tending to 1 for $m(\sigma) \to 0$.

Definition 1.2.2.2. A point $x \in \mathbb{R}^n$ is said to be a generalized Lebesgue point of the function f with respect to the family $\mathfrak{F}$ if

$$\lim_{\sigma\in\mathfrak{F}, m(\sigma)\to 0} \frac{1}{m(\sigma)} \int_{\sigma} f(x-y)dy = f(x).$$

Lemma 1.2.2.3. *If $x \in \mathbb{R}^n$ is a generalized Lebesgue point of a locally bounded function f then it is a generalized Lebesgue point of the function f with respect to any regular family $\mathfrak{F}$.*

Proof. Let $\{\sigma(r)\}$ be a sequence in a regular family $\mathfrak{F}$ such that $\lim_{r\to 0}(m(\sigma(r)) = 0$ and let $B(r)$ be the corresponding family of balls (as in the definition of regular family $\mathfrak{F}$). Then

$$\lim_{r\to 0} \frac{1}{m(\sigma(r))} \int_{\sigma(r)} f(x-y)dy =$$

$$= \lim_{r\to 0} \frac{\int_{B(r)} f(x-y)dy - \int_{B(r)\setminus\sigma(r)} f(x-y)dy}{\int_{B(r)} dy - \int_{B(r)\setminus\sigma(r)} dy} =$$

$$= \lim_{r\to 0} \frac{\frac{1}{m(B(r))}(\int_{B(r)} f(x-y)dy - \int_{B(r)\setminus\sigma(r)} f(x-y)dy)}{1 - \frac{1}{m(B(r))}\int_{B(r)\setminus\sigma(r)} dy} \tag{1.2.2.1}$$

Since the family $\mathfrak{F}$ is regular we have:

$$\lim_{r\to 0} \frac{1}{m(B(r))} \int_{B(r)\setminus\sigma(r)} dy = \lim_{r\to 0} \frac{m(B(r) - m(\sigma(r))}{m(B(r)} \leq$$

$$\leq \lim_{r\to 0} \frac{m(B(r) - c\, m(B(r))}{m(B(r)} = 0.$$

Therefore, using the local boundedness of the function f and equality (1.4.1) we conclude that

$$\lim_{r\to 0} \frac{1}{m(\sigma(r))} \int_{\sigma(r)} f(x-y)dy = f(x)$$

if x is a generalized Lebesgue point of the function f. The lemma is proved. □

Let us try to extend the definition of generalized Lebesgue points for summable functions given on a smooth closed hypersurface $H \subset \mathbb{R}^n$ in the same way as for the usual Lebesgue points. The definition depends on the choice of the volume form ds on H. Of course, there is the natural choice of a volume form as induced by the volume form dy in $\mathbb{R}^n$. However, if H is a hypersurface on a manifold, it is not so.

Definition 1.2.2.4. A point $x \in H$ is said to be a generalized Lebesgue point of a summable function $f \in L^1_{loc}(H)$ if

$$\lim_{r\to 0} \frac{\int_{B(x,r)\cap H} f(y)ds}{\int_{B(x,r)\cap H} ds} = f(x).$$

Lemma 1.2.2.5. *If the function f is locally bounded ($\in L^\infty_{loc}(H)$) then Definition 1.2.2.4 is invariant with respect to a volume form on H.*

Proof. Let $x \in H$ be a generalized Lebesgue point with respect to a volume form ds on H. Any other volume form ds_1 on H has the form $ds_1 = wds$ where w is a positive continuous function on H. Therefore

$$\lim_{r\to 0} \frac{\int_{B(x,r)\cap H} f(y)ds_1}{\int_{B(x,r)\cap H} ds_1} =$$

$$= \lim_{r\to 0} \frac{\int_{B(x,r)\cap H} f(y)(w(y)-w(x))ds + \int_{B(x,r)\cap H} f(y)w(x)ds}{\int_{B(x,r)\cap H}(w(y)-w(x))ds + \int_{B(x,r)\cap H} w(x)ds} =$$

$$= \lim_{r\to 0} \frac{(\int_{B(x,r)\cap H} f(y)(w(y)-w(x))ds)/(w(x)\int_{B(x,r)\cap H} ds)}{(\int_{B(x,r)\cap H} f(y)(w(y)-w(x))ds)/(w(x)\int_{B(x,r)\cap H} ds)+1} +$$

$$+ \lim_{r\to 0} \frac{(\int_{B(x,r)\cap H} f(y))ds)/(\int_{B(x,r)\cap H)} ds)}{(\int_{B(x,r)\cap H} f(y)(w(y)-w(x))ds)/(w(x)\int_{B(x,r)\cap H} ds)+1} = f(x)$$

because of the continuity of the function w and local boundedness of f. □

For summable functions it is easy to find counter-examples.

EXAMPLE 1.2.2.6. Let $H = \mathbb{R}^1$, and $f \in L^1_{loc}(H)$ equal to $y^{-1/3}$ in a neighbourhood of the point $x = 0$ (for $y \neq 0$) and equals to zero if $y = 0$. Then $x = 0$ is a generalized Lebesgue point of the function f with respect to the volume form dy, but it is not with respect to the value form $w(y)dy$ where $w(y) = y^{1/3} + 1$ in a neighbourhood of $y = 0$. $\square$

We note that for the usual Lebesgue points Lemmata 1.2.2.3 and 1.2.2.5 hold true for summable functions.

There is another natural approach to the definition of generalized Lebesgue points of functions given on a smooth hypersurface. Namely, since $H \in C^1$, in a sufficiently small neighbourhood of a point $x \in H$ we can represent the hypersurface H as a graph of a smooth function φ, given on the tangential plane of H in the point x. In this plane there is the volume form $d\hat{s}$ induced from $\mathbb{R}^n$. Then, to a function $f \in L^1_{loc}(H)$, we can naturally associate the function $\hat{f}$ given on the tangential plane and we can consider $x \in H$ as a Lebesgue point of f if it is a Lebesgue point of $\hat{f}$ with respect to the volume form $d\hat{s}$. Using Lemmata 1.2.2.3 and 1.2.2.5 one can show that these 2 approaches (to the definition of the Lebesgue points of functions given on a smooth hypersurface) are equivalent if the function is locally bounded.

Let us give some more close definitions.

Let S be a set of positive ($(n-1)$-dimensional) Lebesgue measure on a smooth closed hypersurface H in $\mathbb{R}^n$.

For points $x \in \mathbb{R}^n$ we denote by $\alpha(x, S)$ the following limit (if it exists):

$$\alpha(x, S) = \lim_{r \to 0} \frac{\int_{B(x,r) \cap S} ds}{\int_{B(x,r) \cap H} ds}. \tag{1.2.2.2}$$

If x is an interior point of S then $\alpha(x, S) = 1$. For boundary points of S it is not so. For example, if the boundary of S is piece-wise smooth then such a limit exists and equals to the value of relative bodily angle of the tangential cone of ∂S in the point x

DEFINITION 1.2.2.7. A point $x \in S$ is said to be a regular point of S if in this point limit (1.2.2.2) exists.

Similar characteristic of the point x we can introduce for the projection S_T of the set S to the tangential plane T in the point x. Namely

$$\alpha(x, S_T) = \lim_{r \to 0} \frac{\int_{B(x,r) \cap S_T} d\hat{s}}{\int_{B(x,r) \cap T} d\hat{s}}.$$

LEMMA 1.2.2.8. *$\alpha(x, S_T) = \alpha(x, S)$, if one of these limits exists.*

PROOF. Without loss of a generality we can consider the situation where $x = 0$ and $T = \{x_n = 0\}$ is the tangential plane to H in this point. Then, in a neighbourhood of zero, the hypersurface H is given by the equality $\rho(y) = y_n - \varphi(y') = 0$, where φ is a smooth function given in a neighbourhood U of zero in T and satisfies $\varphi(0) = 0$, $\varphi(y') = o(|y'|)$ for $|y'| \to 0$, and $y' = (y_1, ..., y_{n-1})$ are coordinates in T. It is clear that $\frac{\partial \varphi}{\partial y_k} \in C(U)$ and $\frac{\partial \varphi}{\partial y_k}(0) = 0$ $(1 \leq k \leq n-1)$. Then the volume form of H is given in the form:

$$ds(y) = *\frac{d\rho(y)}{|d\rho(y)|} = \frac{(-1)^{n-1} dy[n] - \sum_{k=1}^{n-1} (-1)^{k-1} \frac{\partial \varphi}{\partial y_k}(y') dy[k]}{(1 + \sum_{k=1}^{n-1} (\frac{\partial \varphi}{\partial y_k}(y'))^2)^{1/2}}$$

Since $dy_n = d\varphi(y')$ on the hypersurface H, we have

$$ds(y) = -\frac{(-1)^{n-1}dy[n] - \sum_{k=1}^{n-1}(-1)^{n-2}(\frac{\partial\varphi}{\partial y_k}(y'))^2 dy[n]}{(1+\sum_{k=1}^{n-1}(\frac{\partial\varphi}{\partial y_k}(y'))^2)^{1/2}} =$$

$$= (-1)^{n-1}\left(1+\sum_{k=1}^{n-1}\left(\frac{\partial\varphi}{\partial y_k}(y')\right)^2\right)^{1/2} dy' = (-1)^{n-1}b(y')dy'. \tag{1.2.2.3}$$

For sufficiently small $r > 0$ we denote by $P_1(r)$ and $P_2(r)$ the projections to the plane T of the sets $B(1) \cap H/r$ and $B(1) \cap S/r$ correspondingly. It is easy to see that $P_1(r) \subset B(1) \cap T$, $P_2(r) \subset B(1) \cap S_T/r$. Moreover, $P_1(r) = \{y \in T : (y_1^2 + ... + y_{n-1}^2 + \varphi^2(ry)/r^2)^{1/2} \leq 1\}$, and $P_2 = P_1(r) \cap S_T/r$.

We note that the set $P_1(r)$ contains the ball $B(d(r))$ where $d(r) = (1-max(\varphi^2(ry)/r^2)$. Therefore using the above mentioned properties of the function φ and formula (1.2.2.3) we obtain

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\left|\int_{B(r)\cap H} ds - \int_{B(r)\cap T} d\hat{s}\right| \leq \lim_{r\to 0}\left|(-1)^{n-1}\left(\int_{P_1(r)} b(y')dy' - \int_{P_1(r)} dy'\right)\right| +$$

$$+\lim_{r\to 0}\left|(-1)^{n-1}\int_{(B(1)\cap T)\setminus P_1(r)} dy'\right| \leq \lim_{r\to 0} V_{n-1}(1-(d(r))^{n-1}) = 0,$$

where V_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. Hence

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\left|\int_{B(r)\cap H} ds - \int_{B(r)\cap T} d\hat{s}\right| = 0. \tag{1.2.2.4}$$

Arguing similarly we see that

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\left|\int_{B(r)\cap S} ds - \int_{B(r)\cap S_T} d\hat{s}\right| = 0. \tag{1.2.2.5}$$

However $\frac{1}{r^{n-1}}\int_{B(r)\cap T} d\hat{s} = V_{n-1}$, and hence the limits in (1.4) always exists. Now using formula (1.5) we conclude that the existence of the limit $\alpha(x, S)$ implies the existence of the limit $\alpha(x, S_T)$ and contrary. Therefore (1.2.2.4) and (1.2.2.5) imply that $\alpha(x, S) = \alpha(x, S_T)$. The proof is complete. □

At the end of this section let us give one more useful definition.

Definition 1.2.2.9. Let $f \in L^1(\mathbb{R}^n)$. Then we will say that the function

$$\tilde{m}f(x) = \sup_{r>0}\frac{1}{m(B(x,r))}\left|\int_{B(x,r)} f(y)dy\right|$$

is the generalized maximum function of the function f.

It is clear that $\tilde{m}f(x) \leq mf(x)$. Moreover, it is possible that $\tilde{m}f(x) = 0$, but $mf(x) = \infty$.

1.2.3. Some properties of the Poisson integral.

Let $\mathbb{R}^n$ be a boundary hyperplane of the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,\varepsilon) : x \in \mathbb{R}^n,\ \varepsilon > 0\}$. We denote by $\mathcal{P}f(x,\varepsilon)$ the Poisson integral of a summable function f:

$$\mathcal{P}f(x,\varepsilon) = \int_{\mathbb{R}^n} \mathfrak{P}(|x-y|,\varepsilon) f(y)dy = \int_{\mathbb{R}^n} \mathfrak{P}(|y|,\varepsilon) f(x-y)dy,$$

where $\mathfrak{P}(r,\varepsilon) = \frac{2\varepsilon}{\sigma_{n+1}(r^2+\varepsilon^2)^{(n+1)/2}}$ is the Poisson kernel for the half-space $\mathbb{R}^{n+1}_+$ and σ_{n+1} is the area of the unit sphere in $\mathbb{R}^{n+1}$.

For the further discussion we need the following (known) properties of the Poisson kernel.

PROPERTY A. *The kernel $\mathfrak{P}(r,\varepsilon)$ is homogeneous of degree $-n$, i.e. $\mathfrak{P}(\lambda r, \lambda\varepsilon) = \lambda^{-n}\mathfrak{P}(r,\varepsilon)$ for $\lambda > 0$.*

PROPERTY B. $\int_{\mathbb{R}^n} \mathfrak{P}(|y|,\varepsilon)dy = \int_{\mathbb{R}^n} \mathfrak{P}(|z|,1)dz = 1$.

PROPERTY C. $\lim_{r\to 0} r^n\mathfrak{P}(r,1) = 0$, $\lim_{r\to\infty} r^n\mathfrak{P}(r,1) = 0$.

LEMMA 1.2.3.1. *Let $f \in L^1(\mathbb{R}^n)$ then for all $x \in \mathbb{R}^n$ we have:*

$$\sup_{\varepsilon>0} |\mathcal{P}f(x,\varepsilon)| \le \widetilde{m}f(x).$$

PROOF. The proof is similar to the proof of theorem 2 (a) in the book of Stein [St] (p. 77).

We fix a point $x \in \mathbb{R}^n$ and an arbitrary number $\varepsilon > 0$ and set $g(y) = f(x - \varepsilon y)$. Then

$$\mathcal{P}f(x,\varepsilon) = \int_{\mathbb{R}^n} \mathfrak{P}(|y|,1) f(x-\varepsilon y)dy = \int_{\mathbb{R}^n} \mathfrak{P}(|y|,1) g(y)dy = Pg(0,1),$$

$$\widetilde{m}f(x) = \sup_{r>0} \frac{1}{m(B(r))} \left| \int_{B(r)} f(x-y)dy \right| = \sup_{r>0} \frac{\varepsilon^n}{m(B(r))} \left| \int_{B(r/\varepsilon)} f(x-\varepsilon y)dy \right| =$$

$$= \sup_{r>0} \frac{1}{m(B(r/\varepsilon))} \left| \int_{B(r/\varepsilon)} g(y)dy \right| = \widetilde{m}g(0).$$

Thus, it is sufficient to prove that $Pg(0,1) \le \widetilde{m}g(0)$.

If $\widetilde{m}g)(0) = \infty$ the statement is true. Let us suppose then that $\widetilde{m}g(0) < \infty$.

Let $\lambda(r) = \int_{\partial B(1)} g(ry)d\sigma(y)$ where $d\sigma$ is the volume form of the sphere $\partial B(1)$. Then

$$\Lambda(r) = \int_{B(r)} g(y)dy = \int_0^r dt \int_{\partial B(t)} g(y)d\sigma(y) = \int_0^r t^{n-1}\lambda(t)dt.$$

We note that the function $t^{n-1}\lambda(t)$ is summable on the interval $[0,R]$ where $R > 0$, because $g \in L^1(\mathbb{R}^n)$ and

$$\int_0^R |t^{n-1}\lambda(t)|dt \le \int_{B(R)} |g(y)|dy \le \infty.$$

Therefore almost everywhere on $[0, R]$ there is the equality

$$\frac{d\Lambda(r)}{dr} = \frac{d}{dr}\left[\int_0^r t^{n-1}\lambda(t)dt\right] = r^{n-1}\lambda(r).$$

Moreover, since the function $\lambda(r)$ is absolutely continuous (see [RS-N], p.395), we can use the integration by parts.

$$\int_{\mathbb{R}^n} \mathfrak{P}(|y|, 1)g(y)dy = \int_0^\infty \mathfrak{P}(r, 1)r^{n-1}\lambda(r)dr = \lim_{\delta\to 0, N\to\infty} \int_\delta^N \mathfrak{P}(r, 1)r^{n-1}\lambda(r)dr =$$

(1.2.3.1)

$$= \lim_{\delta\to 0, N\to\infty} \int_\delta^N \mathfrak{P}(r, 1)r^{n-1}d\Lambda(r) = \lim_{\delta\to 0, N\to\infty}\left[\mathfrak{P}(r, 1)\Lambda(r)|_\delta^N - \int_\delta^N \Lambda(r)d\mathfrak{P}(r, 1)\right].$$

Because

$$|\Lambda(r)| = \left|\int_{B(r)} g(y)dy\right| \leq m(B(r))\widetilde{m}g(0) = \frac{\sigma_n}{n} r^n \widetilde{m}g(0),$$

using Property C of the Poisson kernel we obtain

$$\lim_{\delta\to 0} \mathfrak{P}(\delta, 1)\Lambda(\delta) \leq \lim_{\delta\to 0} \frac{\sigma_n}{n}\delta^n \widetilde{m}g(0)\mathfrak{P}(\delta, 1) = 0, \tag{1.2.3.2}$$

$$\lim_{N\to 0} \mathfrak{P}(N, 1)\Lambda(N) \leq \lim_{N\to\infty} \frac{\sigma_n}{n} N^n \widetilde{m}g(0)\mathfrak{P}(N, 1) = 0. \tag{1.2.3.3}$$

Now, since $-\frac{d\mathfrak{P}(r,1)}{dr} > 0$ for $r > 0$, formulae (1.2.3.1), (1.2.3.2), and (1.2.3.3) imply that

$$\left|\int_{\mathbb{R}^n} \mathfrak{P}(|y|, 1)g(y)dy\right| = \left|\int_0^\infty \Lambda(r)(-d\mathfrak{P}(r, 1))\right| \leq \int_0^\infty |\Lambda(r)|d(-\mathfrak{P}(r, 1)) \leq$$

$$\leq \frac{\sigma_n}{n}\widetilde{m}g(0)\int_0^\infty r^n d(-\mathfrak{P}(r, 1)) = \widetilde{m}g(0).$$

The proof is complete. □

In the same way as the statement (b) of theorem 1 in [St] (p.237) follows from the statement (a), Lemma 1.2.3.1 implies the following result, mentioned in the paper of Rudin [Ru].

Theorem 1.2.3.2. *If $x \in \mathbb{R}^n$ is a generalized Lebesgue point of a function $f \in L^q(\mathbb{R}^n)$ $(1 \leq q \leq \infty)$ then $\lim_{\varepsilon\to 0} \mathcal{P}f(x, \varepsilon) = f(x)$.*

Proof. First we note that the Poisson integral is well defined for functions in $L^q(\mathbb{R}^n)$ and that $L^q(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ $(1 \leq q \leq \infty)$.

If $x \in \mathbb{R}^n$ is a generalized Lebesgue point of the function f then for any $E > 0$ there is $\delta > 0$ such that for all $0 < r < \delta$ the following inequality holds:

$$\frac{1}{m(B(x, r))}\left|\int_{B(r)} (f(x - y) - f(x))dy\right| < E. \tag{1.2.3.4}$$

We denote by $g(y)$ the function

$$g(y) = \begin{cases} f(y) - fx), \ |y - x| < \delta, \\ 0, \ |y - x| \geq \delta \end{cases}.$$

Obviously, $g \in L^1(\mathbb{R}^n)$. Then

$$|\mathcal{P}f(x, \varepsilon) - f(x)| = \left| \int_{\mathbb{R}^n} \mathfrak{P}(|y|, \varepsilon)(f(x - y) - f(x))dy \right| \leq$$

(1.2.3.5)

$$\leq \left| \int_{|y|<\delta} \mathfrak{P}(|y|, \varepsilon)(f(x - y) - f(x))dy \right| + \left| \int_{|y|\geq\delta} \mathfrak{P}(|y|, \varepsilon)(f(x - y) - f(x))dy \right|.$$

For the first summand in (1.2.3.5) the following estimate holds because of Lemma 1.2.3.1 and (1.2.3.4):

$$\left| \int_{|y|<\delta} \mathfrak{P}(|y|, \varepsilon)(f(x - y) - f(x))dy \right| \leq \left| \int_{|y|<\delta} \mathfrak{P}(|y|, \varepsilon)g(x - y)dy \right| \leq \widetilde{m}g(x) < E.$$

On the other hand

$$\left| \int_{|y|\geq\delta} \mathfrak{P}(|y|, \varepsilon)(f(x - y) - f(x))dy \right| \leq \varepsilon \int_{|y|\geq\delta} \frac{|f(x - y) - f(x)|dy}{(|y|^2 + \varepsilon^2)^{(n+1)/2}} \leq$$

$$\leq \varepsilon \left(\|f\|_{L^q(\mathbb{R}^n)} \||y|^{-n-1}\|_{L^p(\mathbb{R}^n \setminus B(0,\delta))} + |f(x)| \int_{|y|\geq\delta} |y|^{-n-1}dy \right) \leq C^q_\delta(f)\varepsilon$$

where p is the dual number for q (i.e. $1/p + 1/q = 1$). Therefore $\lim_{\varepsilon\to 0} Pf(x, \varepsilon) = f(x)$, which was to be proved. $\square$

Remark 1.2.3.3. For the usual Lebesgue points Theorem 1.2.3.2 holds also in the case where the point (z, ε) tends to the point $(x, 0)$ by any way in a non-tangential cone $K(x)$. For the generalized Lebesgue point it is not true in general.

Let us consider the situation where the integration set is "bad" but not the function f. To formulate the corresponding result we need the notion of the regular point of a set (see Definition 1.2.2.7).

THEOREM 1.2.3.4. *Let S be a set of positive measure in $\mathbb{R}^n$, $f \in C(S) \cap L^q(S)$, $(1 \leq q \leq \infty)$ and x be a regular point of the set S. Then*

$$\lim_{\varepsilon\to 0} \int_S \mathfrak{P}(|x - y|, \varepsilon)f(y)dy = \alpha(x, S)f(x).$$

PROOF. Let us denote by χ_S the characteristic functions of the set S and set

$$g(y) = \begin{cases} \chi_S(y)f(y), \ y \neq x, \\ \alpha(x, S)f(x), \ y = x. \end{cases}$$

It is clear that $g \in L^1(\mathbb{R}^n)$. Because of the continuity of f we have

$$\frac{1}{m(B(x,r))}\int_{B(r)} g(y)dy = \lim_{r\to 0}\frac{f(x)}{m(B(x,r))}\int_{B(x,r)}\chi_s(y)dy+$$

$$+\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)\cap S}(f(x-y)-f(x))dy = \alpha(x,S)f(x).$$

Hence x is a generalized Lebesgue point of the function g.

Now Theorem 1.2.3.2 yields that

$$\lim_{\varepsilon\to 0}\int_S \mathfrak{P}(|x-y|,\varepsilon)f(y)dy = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\mathfrak{P}(|x-y|,\varepsilon)\chi_S(y)f(y)dy =$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\mathfrak{P}(|x-y|,\varepsilon)g(y)dy = \alpha(x,S)f(x).$$

The proof is complete. $\square$

Theorem 1.2.3.4 implies an interesting (from author's point of view) property of the Poisson kernel.

COROLLARY 1.2.3.5. *If x is a regular point of the set $S \subset \mathbb{R}^n$ then*

$$\lim_{\varepsilon\to 0}\int_S \mathfrak{P}(|x-y|,\varepsilon)dy = \alpha(x,S).$$

1.2.4. Theorems on the jump of the Martinelli-Bochner integral.

Let, as in 1.2.1, D be a bounded domain in $\mathbb{C}^n$ and H be a smooth closed hypersurface in D dividing it onto 2 domains D^+ and D^-, and oriented as the boundary of D^-, $\mathfrak{U}$ be the Martielli-Bochner kernel, and Mf be the Martinelli-Bochner integral with a density $f \in L^1(H)$.

THEOREM 1.2.4.1. *If $f \in L^\infty(H)$ and z is a generalized Lebesgue point of f then*

$$\lim_{\varepsilon\to 0}[Mf(z-\varepsilon\nu(z)) - Mf(z+\varepsilon\nu(z))] = f(z),$$

where $\nu(z)$ is the vector of the unit normal to H in z.

PROOF. It is known that the Martinelli- Bochner integral does not depend on unitary transformations (see [Ky]. Hence we can consider only the situation where $z = 0$ and the tangential plane T to H at the point z has the form $T = \{Im\,\zeta_n = 0\}$.

Let $y' = (y_1, ..., y_{2n-1}) \in T$ and $\zeta \in H$. Since H is smooth then it can be represented in a neighbourhood of $z = 0$ in the following way: $\zeta_k = y_k - \sqrt{-1}y_{n+k}$ $(1 \le k \le n-1)$, $\zeta_n = y_n + \sqrt{-1}\varphi(y')$ where φ is a smooth function in a neighbourhood U of zero in the plane T and satisfying the properties mentioned in the proof of Lemma 1.2.2.8.

We note that, for any ball $B(z,R) \subset \mathbb{C}^n$ with $0 < R < \infty$,

$$\lim_{\varepsilon\to 0}\int_{H\setminus B(z,R)}[\mathfrak{U}(z-\varepsilon\nu(z),\zeta) - \mathfrak{U}(z+\varepsilon\nu(z),\zeta)]\,f(\zeta) = 0.$$

Now we choose $R > 0$ such that the hypersurface H is represented as above and denote by $f(y') = f(y', \varphi(y'))$. Then the direct calculations show that

$$\lim_{\varepsilon\to 0}\left[Mf(z-\varepsilon\nu(z)) - Mf(z+\varepsilon\nu(z))\right] =$$

$$= \lim_{\varepsilon\to 0}\sum_{k=1}^{2n-1} c(k)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{y_k\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} - \frac{y_k\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy' +$$

$$+\lim_{\varepsilon\to 0} c(n)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{y_n f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} - \frac{y_n f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy' +$$

$$+\lim_{\varepsilon\to 0} c(n)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{\varphi(y')f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} - \frac{\varphi(y')f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy' +$$

$$+\lim_{\varepsilon\to 0}\sum_{k=1}^{2n-1} c(k)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{\varphi(y')\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} - \frac{\varphi(y')\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy' +$$

$$+\lim_{\varepsilon\to 0}\sum_{k=1}^{2n-1} c(k)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{\varepsilon\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} + \frac{\varepsilon\frac{\partial\varphi(y')}{\partial x_k}f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy' +$$

$$(1.2.4.1) \qquad +\lim_{\varepsilon\to 0}\frac{1}{\sigma_2 n}\int_{B(R)\cap T}\left[\frac{\varepsilon f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} + \frac{\varepsilon f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy'$$

To estimate the summands in the right hand side of (1.2.4.1) we note that

$$(|y'|^2+\varepsilon^2)^{1/2} = |y' \pm \varepsilon\nu(z)| \le |y'-\zeta| + |\zeta \pm \varepsilon\nu(z)| =$$

$$= (|y'|^2 + (\varphi\pm\varepsilon)^2)^{1/2} + |\varphi(|y'|)|.$$

Because $|\varphi(|y'|)| = o(|y'|)$ in $B(R)\cap T$ we see that there exists a constant $C > 0$ such that

$$(1.2.4.2) \qquad C(|y'|^2+\varepsilon^2) \le (|y'|^2 + (\varphi\pm\varepsilon)^2) \ \mathit{for\ all}\ |y'| < R.$$

Also we need the following identity

$$(1.2.4.3) \qquad \frac{1}{a^n} - \frac{1}{b^n} = \left(\frac{1}{a}-\frac{1}{b}\right)\sum_{k=1}^{n-1}\frac{1}{a^k b^{n-k-1}}.$$

Let us use formulae (1.2.4.2) and (1.2.4.3) to estimate the second limit in the right hand side of (1.2.4.1). Since the function f is bounded then using properties of the function φ we conclude that

$$\lim_{\varepsilon\to 0} c(n)\frac{2}{\sigma_2 n}\left|\int_{B(R)\cap T}\left[\frac{y_n f(y')}{(|y'|^2+(\varphi-\varepsilon)^2)^n} - \frac{y_n f(y')}{(|y'|^2+(\varphi+\varepsilon)^2)^n}\right]dy'\right| \le$$

$$\le \lim_{\varepsilon\to 0} c(f)\frac{2}{\sigma_2 n}\int_{B(R)\cap T}\frac{\varphi(y')\varepsilon}{|y'|(|y'|^2+\varepsilon^2)^n}dy' =$$

$$= \lim_{\varepsilon\to 0} c(f) \int_{B(R)\cap T} \frac{\varphi(y')}{|y'|} \mathfrak{P}(|y'|, \varepsilon) dy' = 0$$

Arguing similarly we see that all limits in the right hand side of (1.2.4.1) (except the last one) equal to zero, i.e.

$$\lim_{\varepsilon\to 0} [Mf(z - \varepsilon\nu(z)) - Mf(z + \varepsilon\nu(z))] =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\sigma_2 n} \int_{B(R)\cap T} \left[\frac{\varepsilon f(y')}{(|y'|^2 + (\varphi - \varepsilon)^2)^n} + \frac{\varepsilon f(y')}{(|y'|^2 + (\varphi + \varepsilon)^2)^n} \right] dy' =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\sigma_2 n} \int_{B(R)\cap T} \left[\frac{\varepsilon f(y')}{(|y'|^2 + (\varphi - \varepsilon)^2)^n} - \frac{\varepsilon f(y')}{(|y'|^2 + \varepsilon^2)^n} \right] dy' +$$

$$+ \lim_{\varepsilon\to 0} \frac{1}{\sigma_2 n} \int_{B(R)\cap T} \left[\frac{\varepsilon f(y')}{(|y'|^2 + (\varphi + \varepsilon)^2)^n} - \frac{\varepsilon f(y')}{(|y'|^2 + \varepsilon^2)^n} \right] dy' +$$

$$+ \lim_{\varepsilon\to 0} \int_{B(R)\cap T} \mathfrak{P}(|y'|, \varepsilon) f(y')) dy' = \lim_{\varepsilon\to 0} \int_{B(R)\cap T} \mathfrak{P}(|y'|, \varepsilon) f(y')) dy',$$

because for the first 2 summands we can apply the arguments above.

Finally we see that the point $z = 0$ is a generalized Lebesgue point of the function $f(y') = f(y', \varphi(y')$ (see 1.2.1). Therefore Theorem 1.2.3.2 yelds

$$\lim_{\varepsilon\to 0} \int_{B(R)\cap T} \mathfrak{P}(|y'|, \varepsilon) f(y')) dy' = f(z),$$

which was to be proved. □

Corollary 1.2.4.2. *Let S be a set of positive $(2n-1)$-dimensional measure on H, $z \in S$ be a regular point of the set S, and $f \in C(S) \cap L^1(S)$. Then*

$$\lim_{\varepsilon\to 0} \left[\int_S \mathfrak{U}(z - \varepsilon\nu(z), \zeta) f(\zeta) - \int_S \mathfrak{U}(z + \varepsilon\nu(z), \zeta) f(\zeta) \right] = \alpha(z, S) f(z).$$

Proof. Since the function f is continuous in z, it is bounded in a neighbourhood of this point. Therefore Corollary 1.2.4.2 follows from Theorem 1.2.4.1 as Theorem 1.2.3.4 follows from Theorem 1.2.3.2. □

As in Theorem 1.2.3.2, direction of the tending is very important.

Example 1.2.4.3. Let $S = [0, 1]$ be the interval of the real axis, $H \subset \mathbb{C}^1$ be a smooth curve in the upper half-plane, containing S, $f = 1$ and $z = 0$. It is clear that $\alpha(z, S) = 1/2$. Let the points z^- and z^+ tend to the point $z = 0$ by the lines $arg(z^-) = \beta$ $(0 < \beta < \pi)$, $arg(z^+) = \gamma$ $(\pi < \gamma < 2\pi)$ in such a way that $b|z^-| = |z^+|$ $(b > 0)$. Then the Martinell- Bochner kernel $\mathfrak{U}$ in this case is the Cauchy kernel and

$$\lim_{\varepsilon\to 0} \left[\int_0^1 \mathfrak{U}(z - \varepsilon\nu(z), \zeta) f(\zeta) - \int_0^1 \mathfrak{U}(z + \varepsilon\nu(z), \zeta) f(\zeta) \right] = 1 - \frac{\gamma - \beta}{2\pi} + \frac{\ln b}{2\pi\sqrt{-1}}.$$

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§1.3. Green's integrals and the weak boundary values of solutions of finite order of growth

Let, as in §1.1, P be an operator with injective symbol on an open set $X \subset \mathbb{R}^n$, D be a bounded domain in X with smooth boundary ∂D and $\{B_j\}_{j=0}^{r}$ be a Dirichlet system of order $r \geq p-1$ on ∂D. As we have seen in §1.1 (Proposition 1.1.5), it is convenient to formulate boundary value problems in domain D in terms of $\{B_j\}$ because in this case we do not need to take care about formal agreements between the boundary data. In this way in Chapter 2 we will formulate the Cauchy problem for elliptic systems. For this, however, we need an information on boundary behaviour of solutions of these systems.

In this section we are interested in the weak limit values of the expressions $B_j u$ $(0 \leq j \leq p-1)$ on ∂D of a section $u \in S_P(D)$. Let us distinguish the maximal class of solutions u for which one can speak of these limit values.

We define the function $\rho(x)$ by $\pm dist(x, \partial D)$ where the sign "$-$" corresponds to the case $x \in D$, and "$+$" to the case $x \in X \backslash \overline{D}$. Then, if a neighbourhood U of the boundary $\partial D \in C^m$ $(2 \leq m \leq \infty)$ is sufficiently small, $\rho \in C^m_{loc}(U)$, and $|d\rho| = 1$ in U.

Hence, for small $|\varepsilon|$, the domains $D_\varepsilon = \{x \in D : \rho(x) < -\varepsilon\}$ have boundaries of the class C^m, and as $\varepsilon \to +0(-0)$ they approximate D from inside (outside). Here the unit outward normal vector $\nu(x)$ to the surface ∂D at the point x is given by the gradient $\nabla \rho(x)$. The inner product $ds = \nabla \rho \rfloor dv$ provides the volume form induced by the volume $dv (= dx)$ on X on every surface ∂D_ε.

Definition 1.3.1. *The space $S_{P,B}(D)$ consists of all solutions $u \in S_P(D)$ for which the expressions $B_j u$ $(0 \leq j \leq p-1)$ have weak limit values $u_j \in \mathcal{D}'(F_{j|\partial D})$ on ∂D in the following sense*

$$\lim_{\varepsilon \to 0} \int_{\partial D} < g, B_j u(x - \varepsilon \nu(x)) > ds = \int_{\partial D} < g, u_j > ds \ for\ all\ g \in C^\infty_\circ(F^*_{j|\partial D}).$$

It is clear that, if $u \in S_P(D) \cap C^{p-1}(E_{\overline{D}})$, the weak boundary values of the expressions $B_j u$ $(0 \leq j \leq p-1)$ on ∂D exist and coincide with the usual restrictions $(B_j u)_{|\partial D}$. In order to relate the weak limit values of $B_j u$ $(0 \leq j \leq p-1)$ on ∂D to other (radial, non-tangential, in some norm) limits, Green's formula (as in Theorem 1.1.7) and theorems on the jump of the boundary integral in this formula are usually used. As before, the construction of Green's formula is based on Lemma 1.1.6.

We assume that the complex $\{E^i, P^i\}$ has a fundamental solution in degree 0, say, $\{\mathcal{L}^i\}$, $\mathcal{L}^i \in pdo_{-p_{i-1}}(E^i \to E^{i-1})$ where $pdo_m(E^i \to E^{i-1})$ is the vector space of the all pseudo- differential operators of type $(E^i \to E^{i-1})$ and order m. This means that $\mathcal{L}^{i+1} P^i + P^{i-1} \mathcal{L}^i = 1 - S^i$ on $C^\infty_\circ(E^i)$ where $S^i \in pdo_{-\infty}(E^i \to E^i)$ are smoothing operators, and $S^0 = 0$. In particular, the component $\mathcal{L} = \mathcal{L}^1$ is a left fundamental solution of the differential operator P. This condition holds if, for example, the differential operator $P (= P^0)$ satisfies the Uniqueness Condition $(U)_S$ (see Tarkhanov [T4], Corollary 27.8).

Theorem 1.3.2. *For any solution $u \in S_{P,B}(D)$ we have Green's formula*

$$- \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathcal{L}(x,y), B_j u >_y ds = \begin{cases} u(x), x \in D, \\ 0, x \in X \backslash \overline{D}. \end{cases} \tag{1.3.1}$$

PROOF. First, the theorem of Banach and Steinhaus implies that, for a solution $u \in S_P(D)$, the expressions $B_j u$ $(0 \le j \le p-1)$ have weak limit values $u_j \in \mathcal{D}'(F_{j|\partial D})$ on ∂D if and only if

$$\lim_{\varepsilon \to +0} \int_{\partial D_\varepsilon} < g, B_j u >_x ds = \int_{\partial D} < g, u_j >_x ds \; for\, all\, g \in C_\circ^\infty(F_j^*). \tag{1.3.2}$$

We now choose a number $\varepsilon > 0$ so small that $\partial D_\varepsilon \subset U$. We represent the solution $u \in S_P(D)$ in the domain D by Green's formula, having taken as Green's operator of the differential operator P the operator in Lemma 1.1.6. Then, since the restriction of the differential $d\rho$ on the surface ∂D_ε is equal to zero, we get formula (1.3.1) where in place of D we have the domain D_ε. Having made the limit passage by $\varepsilon \to +0$, and having used equality (1.3.2) we obtain the theorem. □

Formula (1.3.1) gives the apparatus for the effective control of the heuristic consideration that the behaviour of a solution $u \in S_{P,B}(D)$ near a point $x \in \partial D$ in the closure of the domain is completely determined by the "smoothness" property near x on ∂D of the weak boundary values $B_j u$ $(0 \le j \le p-1)$. Thus for $v_j \in \mathcal{D}'(F_{j|\partial D})$ $(0 \le j \le p-1)$ we set $v = \oplus v_j$ so that $v \in \mathcal{D}'(\oplus F_{j|\partial D})$, and

$$\mathcal{G}v(x) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j(y)\mathcal{L}(x,y), v_j >_y ds \; (x \notin \partial D).$$

Let $\mathcal{N}$ be a relatively compact neighbourhood of the point x in X, and $\varphi_\varepsilon \in C_\circ^\infty(X)$ be a function supported in the ε-neighbourhood of $\mathcal{N}$ and beying equal to 1 in $\mathcal{N}$. Then, denoting by χ_D the characteristic function of the domain D, we can rewrite formula (1.3.1) in the form $\chi_D u = \mathcal{G}(\varphi_\varepsilon(\oplus B_j u)) + \mathcal{G}(1-\varphi_\varepsilon)(\oplus B_j u))$. The first summand in (1.3.1) depends only on the values of $B_j u$ $(0 \le j \le p-1)$ in the ε-neighbourhood of the set $\mathcal{N} \cap \partial D$ on the boundary, and the second one is an infinitely differentiable section of E in N. Hence, the character of "the transition" of the solution u from $\mathcal{N} \cap D$ to its weak limit values on $\mathcal{N} \cap \partial D$ is completely determined by the jump behaviour of the surface integral $\mathcal{G}(\varphi_\varepsilon(\oplus B_j u))$ in going across $\mathcal{N} \cap \partial D$. This integral is called Green's integral of the (vector-value) distribution $\varphi_\varepsilon(\oplus B_j u)$.

As an example, let us consider first the situation where the boundary data $B_j u$ are smooth.

Let $x^0 \in \partial D$ be a fixed point and $K(x^0)$ be a non-tangential circular cone with top at the point x^0. Inside of the cone $K(x^0)$ we take 2 points ($x^+ \in X \setminus \overline{D}$ and $x^- \in D$) such that $a|x^+ - x^0| \le |x^- - x^0| \le b|x^+ - x^0|$, where $0 < a \le b < \infty$ are constants.

LEMMA 1.3.3. *Let $\partial D \in C^p$, $u \in C^{p-1}(E_{|\overline{D}})$ be a given section. Then for every multi-index α, with $|\alpha| \le p-1$, there exists the limit*

$$\lim_{x^+, x^- \to x^0} (\partial^\alpha \mathcal{G}(\oplus B_j u)(x^-) - \partial^\alpha \mathcal{G}(\oplus B_j u)(x^+)) = \partial^\alpha u(x^0) \; (x^0 \in \partial D).$$

Moreover, this limit is uniform with respect to $x^0 \in \partial D$ if the cone $K(x^0)$ and constants a, b are fixed.

PROOF. The proof is technically related to the proofs of theorems on the jump behaviour of the Martinelli-Bochner integral (see §1.2), though, of course, it is much more cumbersome. So, we refer readers to Lemma 29.5 (Tarkhanov [T4]). □.

We note that results of this type are some analogues of the Sokhotsky formulae for the Cauchy integral.

LEMMA 1.3.4. *Let $S = \partial D \in C^p$ $(p > 1)$, or C^2 if $p = 1$, $u_j \in C^{p-j-1}$ be summable sections on ∂D. Then the function $\mathcal{G}(\oplus u_j)^+ = \mathcal{G}(\oplus u_j)_{|X\setminus\overline{D}}$ continuously extends to $X\setminus D$ together with its derivatives up to order $p-1$ if and only if the function $\mathcal{G}(\oplus u_j)^- = \mathcal{G}(\oplus u_j)_{|D}$ continuously extends to $\overline{D}$ together with its derivatives up to order $p-1$.*

PROOF. We will use the fact that there exist a smooth function $\hat{u}$ given in a neighbourhood of ∂D in X such that $B_j \hat{u}_{|\partial D} = u_j$ (see Lemma 1.1.5).

If $x^0 \in S$, $\nu(x^0)$ is the unit normal vector to S at the point x^0 and $|\alpha| \le p-1$ then (see Lemma 1.3.3)

$$\lim_{\varepsilon\to 0}(\partial^\alpha \mathcal{G}(\oplus u_j)(x^0 - \varepsilon\nu(x^0)) - \partial^\alpha \mathcal{G}(\oplus u_j)(x^0 + \varepsilon\nu(x^0))) = \partial^\alpha \hat{u}(x^0), \tag{1.3.3}$$

where the limit is uniform on compact subsets in S.

Let, for instance, $\mathcal{G}(\oplus u_j)^-$ continuously extends to $\overline{D}$ together with its derivatives up to order $p-1$. We fix a multi-index $|\alpha| \le p-1$. Then

$$\lim_{\varepsilon\to 0} \partial^\alpha \mathcal{G}(\oplus u_j)(x^0 + \varepsilon\nu(x^0)) = \partial^\alpha \mathcal{G}(\oplus u_j)(x^0) - \partial^\alpha \hat{u}(x^0).$$

Let us define $\mathcal{G}(\oplus u_j)^+$ in the following way:

$$\partial^\alpha \mathcal{G}(\oplus u_j)^+(x) = \begin{cases} \partial^\alpha \mathcal{G}(\oplus u_j)^+(x), \ x \in X\setminus\overline{D}, \\ \partial^\alpha \mathcal{G}(\oplus u_j)^-(x) - \partial^\alpha \hat{u}(x), \ x \in S. \end{cases}$$

Let us show that $\partial^\alpha \mathcal{G}(\oplus u_j)^+$ is continuous in $X\setminus D$. We fix a point $x^0 \in S$ and $E > 0$. Because $\partial^\alpha \mathcal{G}(\oplus u_j)^+$ is continuous on S, there is $\delta_0 > 0$ such that, for $x^1 \in S$ with $|x^1 - x^0| < \delta_0$, we have

$$|\partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1) - \partial^\alpha \mathcal{G}(\oplus u_j)^+(x^0)| < E/2.$$

Decreasing δ_0 (if it is necessary) we can consider $K = \overline{B(x^0, \delta_0)} \cap S$ as a compact subset of S.

Since the hypersurface $S \in C^2$, we can choose $0 < \delta < \delta_0$ in such a way that every point $x \in (X\setminus\overline{D}) \cap B(x^0, \delta)$ is represented in the form $x = x^1 + \varepsilon\nu(x^1)$ where $x^1 \in S$ and $\varepsilon = dist(x, S)$. Then $\varepsilon < \delta$ and $|x^0 - x^1| \le |x^0 - x| + |x - x^1|$, i.e. $x^1 \in K$.

Using the fact that the limit in (1.3.3) is uniform on compact subsets of S and decreasing δ (if it is necessary) we obtain that, for $x^1 \in K$, $0 < \varepsilon < \delta$ the following inequality holds:

$$|\partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1 + \varepsilon\nu(x^1)) - \partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1)| < E/2.$$

Let now $x \in (X\setminus\overline{D}) \cap B(x^0, \delta)$. Then, for some $x^1 \in K$ and $0 < \varepsilon < \delta$ we have $x = x^1 + \varepsilon\nu(x^1)$. Hence

$$|\partial^\alpha \mathcal{G}(\oplus u_j)^+(x^0) - \partial^\alpha \mathcal{G}(\oplus u_j)^+(x)| \le |\partial^\alpha \mathcal{G}(\oplus u_j)^+(x^0) - \partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1)| +$$

$$+ |\partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1 + \varepsilon\nu(x^1)) - \partial^\alpha \mathcal{G}(\oplus u_j)^+(x^1)| < E.$$

Therefore $\mathcal{G}(\oplus u_j)^+$ continuously extends to $X\setminus D$ together with its derivatives up to order $p-1$, if $\mathcal{G}(\oplus u_j)^-$ continuously extends to $\overline{D}$ together with its derivatives up to order $p-1$. The proof is complete. $\square$

COROLLARY 1.3.5. *If for a solution* $u \in S_{P,B}(D)$ *we have* $B_j u \in C^{p-b_j-1}_{loc}(F_{j|\partial D})$ $(0 \le j \le p-1)$ *then* $u \in C^{p-1}(E_{|\overline{D}})$.

PROOF. According to Lemma 1.1.5, we can find a section $\hat{u} \in C^{p-1}_{loc}(E)$ such that $B_j \hat{u} = B_j u$ $(0 \le j \le p-1)$ on ∂D. Then Theorem 1.3.2 and Lemma 1.1.6 imply that $\chi_D u = -\int_{\partial D} G_P(\mathcal{L}(x,y), \hat{u}(y))$. In particular, the integral $\int_{\partial D} G_P(\mathcal{L}(x,y), \hat{u}(y))$, being considered for $x \in X \backslash \overline{D}$, is equal to zero. Therefore it extends continuously together with its derivatives up to order $(p-1)$ to the closure of $X \backslash \overline{D}$. But then Lemma 1.3.4 imply that the integral $\int_{\partial D} G_P(\mathcal{L}(x,y), \hat{u}(y))$ $(x \in D)$ extends continuously together with its derivatives up to order $(p-1)$ to the closure of D. Hence $u \in C^{p-1}(E_{|\overline{D}})$, which which was to be proved. $\square$

In Definition 1.3.1 of the space $S_{P,B}(D)$ we used a Dirichlet system $\{B_j\}$, and it seems that the set of elements of $S_{P,B}(D)$ depends essentially on the choice of this system. The fact that this is not so is unexpected. We shall say that a solution $u \in S_P(D)$ has finite order of growth near the boundary (∂D) if for any point $x^0 \in \partial D$ there are a ball $B(x^0, R)$, and constants $c > 0$ and $\gamma > 0$ such that $|u(x)| \le c\, dist(x, \partial D)^\gamma$ for all $x \in B(x^0, R) \cap D$. In view of the compactness of ∂D, the constants c and γ can be chosen so that the estimate holds for all $x \in \partial D$. The following theorem for harmonic functions was proved by Straube [Stra].

THEOREM 1.3.6. *A solution* $u \in S_P(D)$ *belongs to* $S_{P,B}(D)$ *if and only if it has finite order of growth near* ∂D.

PROOF. Necessity. Any distribution on ∂D locally has finite order of singularity, and the kernel $\mathcal{L}(x,y)$ is infinitely differentiable everywhere outside of the diagonal $\{x = y\}$, and on the diagonal this kernel has the same type of singularity as the well known fundamental solution of $(p/2)$-th degree of the Laplace operator. So the necessity of the condition of the theorem follows from formula (1.3.1).

Sufficiency. Let $u \in S_P(D)$ have finite order of growth, say, γ, near the boundary. It is clear that together with $Pu = 0$ we have $P^*Pu = 0$ where P^* is (formally) adjoint to the differential operator P. The operator P^*P is an determined elliptic operator of order $2p$. We can complete the system $\{B_j\}_{j=0}^{p-1}$ to a Dirichlet system of order $(2p-1)$ on ∂D, say, $\{B_j\}_{j=0}^{2p-1}$, and then we can try to prove that any expression $B_j u$ $(0 \le j \le 2p-1)$ has a weak limit on ∂D according to Definition 1.3.1. When this is proved, we shall have obtained formally more than we require. Of course, it comes to the same thing, because the differential operator P and the system $\{B_j\}_{j=0}^{p-1}$ are arbitrary. So, without loss of a generality, we can require that the differential operator P is determined elliptic. But we can not assume for P^*P the existence of a left fundamental solution (or the condition $(U)_S$ on X). Therefore for P one can only guarantee the existence of a parametrix $\mathcal{L} \in pdo_{-p}(F \to E)$, that is, in particular, $\mathcal{L}P = 1 - S^0$ for some smoothing operator $S \in pdo_{-\infty}(E \to E)$. We now consider this situation. Rojtberg [Roj] showed that one can naturally define a regularization $\hat{u}$ of the solution u as a continuous linear functional on the space $C^{s'}(E_{\overline{D}})$ for a suitable s' depending on the order of singularity of u near the boundary (γ). Then $\hat{u} = u$ in D, and $\hat{u} \in W^{-s,q'}(E_{|D})$ $(= W^{s,q}(E^*_{|D})'))$, where $s > \frac{n}{q} + (\gamma - 1)$, and $\frac{1}{q} + \frac{1}{q'} = 1$ $(q > 1)$. Further, for the solution u there are limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D, these being understood in the following sense. There is a sequence $u^{(\nu)} \in C^\infty(E_{|\overline{D}})$ such that $u^{(\nu)}$ converges to $\hat{u}$ in $W^{-s,q'}(E_{|D})$, and $Pu^{(\nu)}$ converges to zero in $W^{-s-p,q'}(F_{|D})$. Moreover, for any such sequence $u^{(\nu)}$ the sequences $B_j u^{(\nu)}$ $(0 \leq j \leq p-1)$ are fundamental in the spaces $B^{-s-p-b_j-\frac{1}{q'},q'}(F_{j|\partial D})$, and therefore they converge in these spaces to limits u_j. Rojtberg called these sections u_j $(0 \leq j \leq p-1)$ the limit values of the expressions $B_j\hat{u}$ (or, the same, of $B_j u$) on ∂D. Now we want to show that the sections u_j $(0 \leq j \leq p-1)$ are the weak limits of the expressions $B_j u$ in the sense of Definition 1.3.1. To this end we write for the sections $u^{(\nu)}$ Green's formula in the domain D, that is,

$$\chi_D u^{(\nu)} = \mathcal{G}(\oplus B_j u^{(\nu)}) + \mathcal{L}(\chi_D P u^{(\nu)}) + S^0(\chi_D f^{(\nu)})$$

(see, for example, formula (9.13) in the book of Tarkhanov [T5]). If we calculate the limits of the left and right hand side of this equality, for example in the weak topology of the space $\mathcal{D}'(E_{|X\setminus\partial D})$, then we obtain

$$\mathcal{G}(\oplus u_j) + S^0(\chi_D\hat{u}) = \begin{cases} u(x), x \in D, \\ 0, x \in X\setminus\overline{D}. \end{cases} \tag{1.3.4}$$

We have convinced ourselves that the solution u is represented by the limit values on the boundary of the expressions $B_j u$ $(0 \leq j \leq p-1)$ according to Rojtberg [Roj], and by the regularization $\hat{u}$ in D by Green's formula (1.3.4). The second summand on the left hand side of this formula is an infinitely differentiable section of E everywhere on the set X. Therefore the result follows from the following lemma.

Lemma 1.3.7. *We suppose that $D \Subset X$ is a domain with an infinitely differentiable boundary, and $u_j \in \mathcal{D}'(F_{j|\partial D})$ $(0 \leq j \leq p-1)$ are given sections on ∂D. Then, for all sections $g_j \in \mathcal{D}(F^*_{j|\partial D})$ $(0 \leq j \leq p-1)$ we have*

$$\lim_{\varepsilon\to+0}\int_{\partial D} < g_j, B_j(\mathcal{G}(u))(x-\varepsilon\nu(x)) - B_j(\mathcal{G}(u))(x+\varepsilon\nu(x)) >_x ds =$$

$$= \int_{\partial D} < g_j, u_j >_x ds. \tag{1.3.5}$$

Proof. We fix a section $g_j \in \mathcal{D}(F^*_{j|\partial D})$ and we find a section $g \in C^\infty_{loc}(F^*)$ such that $C_j g = g_j$, and $C_j g = 0$ for $i \neq j$ on ∂D. It is not difficult to construct such a section g, for example, using the formulae for the jumps in crossing ∂D of Green's type integral with a smooth density. Then using Lemma 1.1.6 we can write

$$\lim_{\varepsilon\to+0}\int_{\partial D} < g_j, [B_j(\mathcal{G}(u))(x-\varepsilon\nu(x)) - B_j(\mathcal{G}(u))(x+\varepsilon\nu(x))] >_x ds =$$

$$= \lim_{\varepsilon\to+0}\left[\int_{\partial D_\varepsilon}\sum_{j=0}^{p-1} < C_j g, B_j(\mathcal{G}u) >_x ds - \int_{\partial D_{-\varepsilon}}\sum_{j=0}^{p-1} < C_j g, B_j(\mathcal{G}u) >_x ds\right] =$$

$$= \lim_{\varepsilon\to+0}\int_{\partial(D_{-\varepsilon}\setminus D_\varepsilon)} G_P(g, \mathcal{G}u).$$

Repeating the arguments given on p.291 in the book of Tarkhanov [T4] we obtain that the last limit exists, and that it is equal to

$$\int_{\partial D} < C_j g, u_j >_x ds = \int_{\partial D} < g_j, u_j >_x ds,$$

which was to be proved. □

As one can see from the proof of Lemma 1.3.7, it holds also for a domain D with a boundary of finite, perhaps, very high degree of smoothness. The same can be applied to the smoothness of the sections g_j in (1.3.5). These depend on the orders of singularity of the given sections u_j $(0 \le j \le p-1)$ which are finite since the surface ∂D is compact.

We can now complete the proof of Theorem 1.3.6. In fact, if $g \in \mathcal{D}(F^*_{j|\partial D})$ where $0 \le j \le p-1$, then, from formula (1.3.4) and Lemma 1.3.7, we obtain

$$\lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j u(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j(\mathcal{G}(\oplus u_j))(x - \varepsilon\nu(x) - B_j(S^0(\chi_D \hat{u})(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j(\mathcal{G}(\oplus u_j))(x - \varepsilon\nu(x)) - B_j(S^0(\chi_D \hat{u})(x + \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j(\mathcal{G}(\oplus u_j))(x - \varepsilon\nu(x)) - B_j(\mathcal{G}(\oplus u_j))(x + \varepsilon\nu(x)) >_x ds =$$

$$= \int_{\partial D} < g_j, u_j >_x ds,$$

that is, $u \in S_{P,B}(D)$. Hence Theorem 1.3.6 is completely proved. □

We note that Lemma 1.3.7 is similar to the theorem on the weak jump of the Bochner - Martinelli integral which was proved by Chirka [Ch].

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§1.4. Weak values of solutions in $L^q(D)$ on the boundary of D

Again let P be a differential operator with an injective symbol on X, not necessarily satisfying assumptions of §1.3, and u be a solution of the system $Pu = 0$ in D of Lebesgue class $L^q(E_{|D})$ where $1 \le q \le \infty$. What can one say of the limit values on ∂D of the expressions $B_j u$ $(0 \le j \le p-1)$? Extrapolating the situation for holomorphic functions one can say that the class of solutions in $S_P(D) \cap L^q(E_{|D})$ is wider than the so-called Hardy class $H^q_{P,B}(D)$ which consists of all solutions $u \in S_{P,B}(D)$ whose weak limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ belong to $L^q(F_{j|\partial D})$. Moreover, a priori it is not clear, whether the solution $u \in S_P(D) \cap L^q(E_{|D})$ has finite order of growth near ∂D, that is whether the expressions $B_j u$ $(0 \le j \le p-1)$ have weak limit values on ∂D. Estimates of growth near ∂D of solutions $u \in L^2(F_{|D})$ could be obtained from the asymptotic behaviour of the reproducing kernel of the domain D with respect to the Hilbert space $L^2(F_{|D})$. However even in the case of the Cauchy-Riemann system this asymptotic behaviour is not known for all domains (see Henkin [He], p.68). In this section we prove that for any solution $S_P(D) \cap L^1(E_{|D})$ there are weak limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on the boundary. Then the theorem of Rojtberg [Roj] allows us to know the smoothness of these values on ∂D.

So, we fix $u \in S_P(D) \cap L^q(E_{|D})$, where $1 \le q \le \infty$, and a number j $(0 \le j \le p-1)$. Putting aside for the meanwhile the questions of the correctness of the definition, we associate a vector-valued distribution $u_j \in \mathcal{D}'(F_{j|\partial D})$ with the solution u in the following way. Let $g_j \in C^{b_j+1}(F^*_{j|\partial D})$. Using Lemma 1.1.5, we find a section $g \in C^p_{loc}(F^*)$ such that $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D. Then we set

$$(1.4.1) \qquad < g_j, u_j >= - \int_D < P'g, u >_x dv \quad (g_j \in C^{b_j+1}(F_{j|\partial D})$$

LEMMA 1.4.1. *Definition (1.4.1) is correct, that is, it does not depend on the choice of the section $g \in C^p_{loc}(F^*)$ for which $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D.*

PROOF. It is sufficient to show that, if for a section $g \in C^p_{loc}(F^*)$ the boundary values on ∂D of the expressions $C_j g$ $(0 \le j \le p-1)$ are equal to zero, then $\int_D < P'g, u > dv = 0$.

First of all we replace the section g by another section with the same differential $P'g$, and with derivatives up to order $(p-1)$ are equal to zero on ∂D. For this we represent the section g in D by means of the homotopy formula on a manifold with boundary (see, for example, Tarkhanov [T5], (9.13)). Bearing in mind the connection between Green's operators of the differential operator P and the transposed of P (see Proposition 1.1.3), and using Lemma 1.1.6 we have

$$(1.4.2) \qquad \mathcal{L}'(\chi_D P'g) + P^{1'} \mathcal{L}'(\chi_D g) + \mathcal{S}^{1'}(\chi_D g) = \chi_D g.$$

Let $v \in W^{2p,\widetilde{q}}(E^{2^*})$ (where $\widetilde{q} >> 1$) be an extension of the section $\mathcal{L}(\chi_D g)$ from $X \backslash D$ to the whole set X. The number $\widetilde{q}$ can be chosen as large as we want, however for our purposes it is sufficient that $\widetilde{q} > n$, and $\widetilde{q} \ge q'$ where q' is dual to the index q, that is, $1/q + 1/q' = 1$. Then, if we consider the section $\widetilde{g} = \mathcal{L}(\chi_D P'g) + P^{1'} v + S^{1'}(\chi_D g)$, we can say that $g \in W^{p,\widetilde{q}}(F^*)$, and $P'\widetilde{g} = P'g$. Moreover, from formula (1.4.2), $\widetilde{g} \equiv 0$ outside of D, but since $\widetilde{g} \in C^{p-1}_{loc}(F^*)$ we have $D^\alpha \widetilde{g} = 0$ $(|\alpha| \le p-1)$ on ∂D. Then, replacing if necessary g by $\widetilde{g}$, we assume without loss of generality that the derivatives of g up to order $(p-1)$ vanish on ∂D. In this case there is some loss of smoothness of g, but this is not important for us. Further, we use the lemma of Bochner which says that for any $\varepsilon > 0$ there is a function $\varphi_\varepsilon \in \mathcal{D}(X)$ $(0 \le \varphi_\varepsilon \le 1)$ with support in the ε-neighbourhood of the boundary ∂D which is equal to unit in some smaller neighborhood of ∂D, and for which $|D^\alpha \varphi_\varepsilon| \le c_\alpha \varepsilon^{-|\alpha|}$ everywhere in $\mathbb{R}^n$ where the constant c_α does not depend on ε (see Hörmander [Hö1], theorem 1.4.1). We have

$$(1.4.3) \quad \int_D < P'g, u >_x dv = \int_D < P'(1-\varphi_\varepsilon)g, u >_x dv + \int_D < P'(\varphi_\varepsilon g), u >_x dv$$

Since the section $(1-\varphi_\varepsilon)$ has compact support in D then, from Stokes' formula, the first summand on the right hand side of (1.4.3) disappears. As for the second summand we can write

$$\int_D < P'(\varphi_\varepsilon g), u >_x dv =$$

$$= \sum_{|\alpha|\le p} (-1)^{|\alpha|} \sum_{|\beta|\le|\alpha|} \binom{\alpha}{\beta} \int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), u >_x dv. \tag{1.4.4}$$

We want to prove that the right hand side converges to zero, as $\varepsilon \to +0$. For to do this it is suffices to estimate a typical summand in (1.4.4): $\int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), u >_x dv$ ($\beta \ne 0$). Having used the Hölder inequality, and taking into consideration the estimates of the derivatives of the function φ_ε we obtain with a constant $c > 0$ which does not depend on ε such that

$$\left| \int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), u >_x dv \right| \le$$

$$\le \|D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g)\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \|u\|_{L^q(E_{D\setminus D_\varepsilon})} \le$$

$$\le c_1 \varepsilon^{-|\beta|} \|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \|u\|_{L^q(E_{D\setminus D_\varepsilon})} \tag{1.4.5}$$

Since $g \in C^{p-1}_{loc}(F^*)$, and $D^\gamma g = 0$ ($|\gamma| \le p-1$) on ∂D, using the localization process and the repeated use of the Newton-Leibniz formula, it is not difficult to see there is a constant $c_2 > 0$ such that for all sufficiently small $\delta > 0$ we have

$$\|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|\partial D_\delta})} \le c_2 \delta^{p-1-|\alpha|+|\beta|+1/q} \|g\|_{W^{p,q'}(F^*_{|D\setminus D_\delta})} \tag{1.4.6}$$

Similar considerations can be found in the book of Mihailov [Mi] (p.148). Now we choose $\varepsilon > 0$ sufficiently small and integrate inequality (1.4.6) with respect to δ from 0 to ε. Then using the Fubini theorem we obtain the inequality

$$\|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \le c'_2 \varepsilon^{p-|\alpha|+|\beta|+1/q} \|g\|_{W^{p,q'}(F^*_{|D\setminus D_\varepsilon})}$$

where $c'_2 = c_2/((p-1-|\alpha|+|\beta|+1/q)q'+1)^{1/q'}$. Substituting this estimate in (1.4.5), we obtain

$$\left| \int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), u >_x dv \right| \le$$

$$\le c_1 c'_2 \varepsilon^{p-|\alpha|+|\beta|+1/q} \|g\|_{W^{p-|\alpha|,q'}(F^*_{|D\setminus D_\varepsilon})} \|u\|_{L^q(E_{D\setminus D_\varepsilon})},$$

So we can find a constant $c > 0$ depending only on the norms of the coefficients of the differential operator P in the domain D such that for all sufficiently small $\varepsilon > 0$ we have

$$\left| \int_D < P' g, u >_x dv \right| \le c \|g\|_{W^{p,q'}(E^*_{|D\setminus D_\varepsilon})} \|u\|_{L^q(E_{D\setminus D_\varepsilon})} \tag{1.4.7}$$

The property of the absolute continuity of a Lebesgue integral with respect to a domain of integration implies that for any q in the range $1 \le q \le \infty$ the expression on the right hand side of (1.4.7) converges to zero as $\varepsilon \to +0$. Therefore $\int_D < P'g, u >_x dv = 0$, which proves the lemma. $\square$

As one can see, if $q = 1$ in the proof of Lemma 1.4.1 the arguments fail. Thus in this case the definition (1.4.1) needs some modification. Namely, it is necessary to change the smoothness of the sections g_j in (1.4.1) by "+0", that is, we must take, for example, $g \in C^{b_j+1,\lambda}(F^*_{j|\partial D})$, where $\lambda > 0$.

The distributions $u_j \in \mathcal{D}'(F_{j|\partial D})$ $(0 \le j \le p-1)$ constructed in (1.4.1) we now take as the weak limit values of the expressions $B_j u$ on ∂D. It is clear that if $u \in C^{p-1}(E_{|\overline{D}})$ then u_j is simply the pointwise restriction of $B_j u$ on ∂D. However in the general case the identification of u_j $(0 \le j \le p-1)$ with the weak limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D by definition (1.4.1) is difficult. Later on we shall show that this identification is valid, but now we begin with the justification of the naturality of definition (1.4.1).

LEMMA 1.4.2. *For any solution $u \in S_P(D) \cap L^q(E_{|D})$ $(1 < q \le \infty)$ the following Green's formula holds:*

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j g, B_j u >_x ds = -\int_D < P'g, u >_x dv \quad (g \in C^p(F^*_{|\overline{D}})). \tag{1.4.8}$$

PROOF. For each number $1 \le j \le p-1$ we construct a section $g^{(j)} \in C^p_{loc}(F^*)$ such that $C_j g^{(j)} = C_j g$, and $C_i g^{(j)} = 0$ for $i \ne j$ on ∂D. We set $g_0 = g - g^{(1)} - \ldots - g^{(p-1)}$. Then $g_0 \in C^p_{loc}(F^*_{\overline{D}})$, $C_0 g^{(0)} = C_0 g$, and $C_i g^{(0)} = 0$ for $i \ne 0$ on ∂D. Hence, according to definition (1.4.1) we can write

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j g, B_j u >_x ds = \sum_{j=0}^{p-1} \left(-\int_D < P' g^{(j)}, u >_x dv \right) =$$

$$= -\int_D < P'g, u >_x dv,$$

which was to be proved. $\square$

Formula (1.4.8) holds also for solutions $u \in S_P(D) \cap L^1(E_{|D})$, however with sections g whose smoothness is greater than "+0", that is, for $g \in C^{p,\lambda}(F^*)$ where $\lambda > 0$.

LEMMA 1.4.3. *For any solution $u \in S_P(D) \cap L^1(E_{|D})$ Green's formula (1.3.1) holds.*

PROOF. Let x be a fixed point belonging to $X \backslash \partial D$. We take some function $\varphi \in \mathcal{D}(X)$ which is equal to 1 in a neighbourhood of ∂D, and vanishes on some neighborhood of the point x. It is clear that $\varphi \mathcal{L} \in C^\infty_{loc}(E_x \otimes F^*)$, therefore formula (1.4.8) implies that

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathcal{L}, B_j u >_x ds = -\int_D < P'(\varphi \mathcal{L}, u >_x dv. \tag{1.4.9}$$

We choose $\varepsilon > 0$ so small that $\varphi \equiv 1$ in some neighbourhood of "the piece" $D \backslash D_\varepsilon$. Since $P'\mathcal{L}(x, .) = 0$ everywhere outside of the point x, it follows that the integral on the right hand side of formula (1.4.9) is equal to the similar integral taken over the domain D_ε. But $u \in S_P(\overline{D}_\varepsilon)$, therefore the last integral is equal to $-\int_{\partial D_\varepsilon} G_P(\mathcal{L}(x,.), u)$, that is, $(\chi_D u)(x)$, which was to be proved. $\square$

We can now formulate the principal result of this section. As before, we denote by $B^{s,q}(F_{j|\partial D})$ the usual Besov spaces of sections of the bundles F_j over ∂D (see Kudrjavtsev and Nikolskii [KdNi]). In particular, if s is not an integer or $q = 2$ then $B^{s,q}(F_{j|\partial D}) = W^{s,q}(F_{j|\partial D})$. If $1 < q < \infty$ then in definition (4.1) we can take $g_j \in B^{b_j+1/q',q}(F_{j|\partial D})$ $(0 \le j \le p-1)$. Lemma 2.2 from the paper of Rojtberg [Roj] guarantees existence of a section $g \in W^{s,q}(F^*_{|\partial D})$ such that $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D. Then one can substitute g into the right part of (1.4.1). Moreover, the above-mentioned lemma of Rojtberg [Roj] says that the mapping $g_j \to g$ is continuous. Using Hölder's inequality it is easy to conclude that $B_j u \in B^{-b_j-1/q',q'}(F_{j|\partial D})$ $(0 \le j \le p-1)$ (see our paper [ShT4]). However we obtain a more general result directly from the fundamental theorem of Rojtberg [Roj].

THEOREM 1.4.4. *For a solution $u \in S_P(D) \cap L^1(E_{|D})$ the limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D defined by formula (1.4.1) are the weak limit values. Moreover $u \in W^{s,q}(E_{|D})$ $(1 < q < \infty)$ if and only if $B_j u \in B^{s-b_j-1/q,q}(F_{j|\partial D})$ $(0 \le j \le p-1)$.*

PROOF. Again we shall try to reduce the proof to the corresponding fact for solutions of elliptic systems. We fix a section $u \in S_P(D) \cap L^q(E_{|D})$, $q > 1$, satisfying $Pu = 0$ in D. Then u must also satisfy $\Delta u = 0$ where $\Delta = P^*P$ is an (determined) elliptic differential operator of type $E \to E$, and of order $2p$ on X. The system $\{B_j\}_{j=0}^{p-1}$ can be replaced with a Dirichlet system of order $(2p-1)$ on ∂D in the following way. We set $\widetilde{B_j} = B_j$ for $0 \le j \le p-1$, and $\widetilde{B_j} = *^{-1} C_{j-p} * P$ for $p \le j \le 2p-1$. Then $\{\widetilde{B_j}\}_{j=0}^{2p-1}$ is a Dirichlet system of order $(2p-1)$ on ∂D, and the Dirichlet system $\{\widetilde{C_j}\}_{j=0}^{2p-1}$ corresponding to it by Lemma 1.1.6 (with $P = \Delta$) has the form $\widetilde{C_j} = -C_j * P *^{-1}$ for $0 \le j \le p-1$, and $\widetilde{C_j} = - * B_{j-p} *^{-1}$ for $p \le j \le 2p-1$. We now use a relation (which is similar to (1.4.1) to define the limit values of the expressions $\widetilde{B_j} u$ $(0 \le j \le 2p-1)$ on ∂D in our new situation. More precisely, these expressions are only interesting for $(0 \le j \le p-1)$. So, let $g \in C^{b_j+1}(F^*_{j|\partial D})$ $(0 \le j \le p-1)$. Using Lemma 1.1.5 we find a section $\mathfrak{G} \in C^{2p}_{loc}(E^*)$ such that $C_j * P *^{-1} \mathfrak{G} = g$, and $\widetilde{C_i}\mathfrak{G} = 0$ for $i \ne j$ $(0 \le i \le 2p-1)$ on ∂D. Then we set

$$< g, B_j u >= -\int_D < \Delta' \mathfrak{G}, u >_x dv, \quad (g_j \in C^{b_j+1}(F^*_{j|\partial D})). \tag{1.4.10}$$

However, if we define $B_j u$ on ∂D by means of formula (1.4.1), the choice of g in Lemma 1.4.1 is unimportant. In particular, nothing prevents us from taking $g = *P *^{-1} \mathfrak{G}$ in (1.4.1). Then we obtain equality (1.4.10). Hence the definition of the limit values of $B_j u$ $(0 \le j \le p-1)$ on ∂D does not depend on whether u is a solution of the system $Pu = 0$ or $\Delta u = 0$. So, replacing the operator P by Δ we may suppose without loss of a generality that P is elliptic. But then the first part of Theorem 1.4.4 follows from Lemmata 1.4.3 and 1.3.7. For, from Lemma 1.4.3, the solution u is represented by the limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D which are defined in accordance with equality (1.4.1) by means of Green's formula (1.3.1). And Lemma 1.3.7 asserts that the weak jump in going across ∂D of the expressions $B_j \mathcal{G}(\oplus B_i u)$ $(0 \le j \le p-1)$ coincides with $B_j u$. Hence the limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D exist, and they coincide with the limit values calculated by the formula (1.4.1). This proves the first part of the theorem for solutions $u \in L^q(E_{|D})$ $(q > 1)$, and for $q = 1$ we must make obvious modifications. To prove the second part of the theorem we assume in addition that $u \in S_P(D) \cap W^{s,q}(E_{|D})$ where $1 < q < \infty$. Rojtberg [Roj] proved that there are limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D in the following sense. There is a sequence $u^{(\nu)} \in C^\infty(E_{|\overline{D}})$ such that $u^{(\nu)}$ converges to u in $W^{s,q}(E_{|D})$ and Pu converges to zero in $W^{s-p,q}(F_{|D})$. Moreover, for any such a sequence $u^{(\nu)}$ the sequence $B_j u^{(\nu)}$ $(0 \le j \le p-1)$ is fundamental in Besov space $B^{s-b_j-1/q,q}(F_{j|\partial D})$, and therefore it converges in this space to a limit u_j. Arguing as in the proof of Theorem 1.3.6 we see that the solution u is represented by the boundary values u_j by means Green's formula (1.3.4). Then Lemma 1.3.7 again shows that the sections u_j $(0 \le j \le p-1)$ are the limit values on ∂D of the expressions $B_j u$. So the weak limit values of the expression $B_j u$ $(0 \le j \le p-1)$ on ∂D belong to the Besov space $B^{s-b_j-1/q,q}(F_{j|\partial D})$.

Conversely, if such an inclusion holds then formula (1.3.1) and the theorems on boundedness of potential (or co-boundary) operators on a manifold with boundary (see Rempel and Schulze [ReSz], 2.3.2.5) imply that $u \in W^{s,q}(E_{|D})$. This proves Theorem 1.4.4. $\square$

This theorem, in particular, shows that for a solution $u \in S_P(D) \cap L^1(E_{|D})$ definition (1.4.1) of the boundary values $B_j u$ $(0 \le j \le p-1)$ on ∂D does not depend on the choice of the differential operator P. æ

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CHAPTER II

GREEN'S INTEGRALS AND BASES WITH DOUBLE ORTHOGONALITY IN THE CAUCHY PROBLEM FOR ELLIPTIC SYSTEMS

§2.0. Introduction

We shall consider in this chapter the Cauchy problem for solutions of a differential equation $Pu = 0$ where $P \in do_p(E \to F)$ is a differential operator of order $p \geq 1$ with an injective symbol on an open set $X \subset \mathbb{R}^n$. As above, here $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X.

In the case of overdetermined elliptic systems (i.e. for $l > k$), similarly to the Cauchy-Riemann system in several complex variables, under sufficiently broad assumptions about the differential operator P, it is possible to include it into some elliptic complex of differential operators on X, say, $\{E^i, P^i\}$ where $E^i = X \times C^{k_i}$ are (trivial) vector bundles over X which are different from zero only for $0 \leq i \leq N$, and $P^i \in do_{p_i}(E^i \to E^{i+1})$ where $P^0 = P$ (see Samborskii [Sa]). We shall often use this identification, assuming that the conditions on P are fulfilled.

Throughout of this chapter we assume that P satisfies the weak unique continuation principle:

$\mathbf{(U)}_{\boldsymbol{S}}$ *if for a domain $O \subset X$ we have $Pu = 0$ in O, and $u = 0$ on a non-empty open subset of O then $u \equiv 0$ in O.*

The Cauchy problem we are interested in is roughly formulated as follows:

Problem 2.0. *Let D be a subdomain of X and S be a subset of ∂D of positive $(n-1)$-dimensional measure. Let u_α ($|\alpha| \leq p-1$) be given sections of E over S. It is required to find a solution $u \in S_P(D)$ whose derivatives $D^\alpha u$ up to order $(p-1)$ have, in a suitable sense, limit values $(D^\alpha u)_{|S}$ on S such that $(D^\alpha u)_{|S} = u_\alpha$ ($|\alpha \leq p-1$).*

Later on we will state Problem 2.0 in a more correct way.

The plan of the chapter is the following.

In §2.1 we elaborate the operator-theoretical foundations for applying bases with double orthogonality to the problem of the continuation of classes of functions from massive subsets to the whole set. In a paper dated 1927 Bergman (see [Brg], p.14-20) developed this remarkable concept considering sequence of analytic functions which are orthogonal with respect to L^2-scalar product on couples of domains one of which contains the closure of the other. His aim was the study of criteria for analytic continuation. This beautiful and potentially useful idea did not receive sufficient recognition, probably because its practical application requires to solve preliminarily an eigenvalue problem, which may turn out to be quite difficult to solve. Bases with double orthogonality appeared again in a series of the papers by

Slepian and Pollak [SlPo], Landau and Pollak [LPo1], [LPo2], and Slepian [Sl]) in the sixties somehow independently of Bergman. Shapiro [Shp1] is sure that Bergman knew well that the phenomenon of double orthogonality had a more general scope going far beyond the problem of analytic continuation in complex analysis.

The fuctional analysis involved in the study of bases with the double orthogonality property reduces essentially to the spectral theorem for a compact self-adjoint operator, which is traditionally credited to F. Riesz (see Riesz and Sz.- Nagy [RS-N], s. 93). Krasichkov [Kra] has shown how the spectral theorem leads quite simply to an abstract Bergman theorem about the existence of bases with double orthogonality (see also Shapiro [Shp1], [Shp2]). Our account in §2.1 reproduces Bergman's concept in general, except that we consider continuous systems of functions with double orthogonality.

As the Cauchy Problem 2.0 may be unsolvable even in the class of all smooth (vector) functions u in D (not only those satisfying $Pu = 0$) there are formal difficulties in the setting of the problem. To remove these difficulties it is necessary that the sections $u_\alpha (|\alpha| \leq p-1)$ should be restrictions to S of the corresponding derivatives of some smooth section in D. This is connected with the correct setting of the Cauchy problem which corresponds to a suitable Green's formula for solutions. In §2.2 we formulate the Cauchy problem in a more correct way and indicate a rather general situation where it has no more then one solution.

In §2.3 a solvability criterion for the Cauchy problem for elliptic systems in the Hardy class $H^2_{P,B}(D)$ (see Tarkhanov [T2]) is deduced in terms of bases with double orthogonality on the boundary of D. The corresponding eigenvalue problem is associated with a non-compact operator. Surface bases with double orthogonality are continuous systems of generalized eigenvectors of this operator (see Berezanskii [Bz], ch. V). Surface bases with double orthogonality in the Cauchy problem for holomorphic functions of one variable seemed to have been first applied by Krein and Nudelman [KrNu].

In §2.4 we prove a solvability criterion for the Cauchy problem for elliptic systems in terms of a Green's integral. Using the Cauchy data on S we construct a Green's integral satisfying $Pu = 0$ everywhere outside of S. Then the Cauchy problem is solvable if and only if this integral can be continued across S from the complement of D as a solution of the system $Pu = 0$ $(\in W^{s,q}(E_{|D}))$. Although it is possible to obtain interesting applications directly from this observation, this result has an auxiliary character. In spite of the simplicity of the idea, its proof is complicated by the necessity of using nontrivial results from the theory of pseudo-differential operators on manifolds with boundary. For instance, we need to use a theorem on the boundedness in Sobolev spaces of potential operators which was recently proved (see Eskin [Es], Rempel and Schulze [ReSz] and others).

In §2.5 the extendibility condition (as a solution of the system $Pu = 0$) across S of the Green's integral is expressed in terms of space bases with double orthogonality. Its construction is connected with the solution of an eigenvalue problem for a compact operator, so this part of the application of bases with double orthogonality is very similar to the original concept of Bergman [Brg]. We note that these ideas were first tested on the Cauchy problem for holomorphic functions (see [ShT4]) and we found some hints in the considerations of Aizenberg and Kytmanov [AKy].

The use of bases with double orthogonality not only gives information about solvability conditions for the Cauchy problem, but leads to explicit formulae for its solutions. A Carleman function of the Cauchy problem for solutions of elliptic systems is constructed in §2.6.

In §2.7 we describe a stability set in the Cauchy problem for elliptic systems.

In §§2.8, 2.9, as examples, we consider the Cauchy problem for the Laplace equation and for the Lamé type system in $\mathbb{R}^n$.

In §2.10 we show how the Cauchy problem for overdetermined elliptic systems with real analytic coefficients may be reduced to the Cauchy problem for solutions of determined elliptic systems which was considered in sections §§2.4-2.8.

In §2.11 we prove a solvability criterion for the Cauchy problem for systems with injective symbol in terms of a Green's integral. By using "Cauchy data" on S we construct a Green's integral which satisfies $P^*Pu = 0$ everywhere outside an arbitrary small neighbourhood of S on ∂D. Then the Cauchy problem is solvable if and only if this integral analytically extends across S from the complement of D to this domain in a suitable Sobolev class, and the Cauchy data on S satisfy the tangential equation on S.

In §2.12 the condition for extendibility (as a solution of the system $P^*Pu = 0$) across S of Green's integral is written in terms of space bases with double orthogonality. As in §2.5, their construction depends on the solution of an eigenvalue problem for a compact self- adjoint operator.

Again the use of bases with double orthogonality not only gives information about solvability conditions for the Cauchy problem but also leads to explicit formulae for its solutions. A Carleman function of the Cauchy problem for solutions of systems with injective symbols is constructed in §2.13.

Finally, in §2.14 we consider some examples of differential equations of the simplest type including the Cauchy-Riemann system in several complex variables. These are systems of first order differential equations which are matrix factorizations of the Laplace operator. A system of homogeneous polynomials in $\mathbb{R}^n$ possessing the double orthogonality property relative to integration over every ball centered at zero is constructed. Using it we obtain the solvability condition in an explicit form and obtain a formula for the regularization of the Cauchy problem for the matrix factorizations of the Laplace operator in this special case. More exactly, S is a smooth hypersurface and D is the one of the two domains in which S divide a ball B centered at 0 which does not contain the origin. The theorems on the solvability of the Cauchy problem and on the Carleman formula for holomorphic functions of one variable obtained in this way are among the simplest ones (see [Aky] and [A]). For the Cauchy-Riemann system in several complex variables, the corresponding results were obtained in [AKy] and [ShT4]. æ

§2.1. Bases with double orthogonality

As Shapiro [Shp1] has observed, Bergman's problem is a special case of the question of when a given element of a Hilbert space belongs to the image of some injective compact operator with dense image.

In practice this problem appears usually in the following way. There is some linear continuous mapping of Hilbert spaces, $T : H_1 \to H_2$, say. Further, in H_1 a closed subspace

Σ_1 is distinguished by some considerations. It is very helpful when the image of Σ_1 by the mapping T is closed in H_2. However this is not usually the case. In any case we denote by Σ_2 the closure of this image. Hence Σ_2 also is a Hilbert space with the Hermitian structure induced from H_2.

PROBLEM 2.1.1. *Let $h_2 \in \Sigma_2$. It is required to find a vector $h_1 \in \Sigma_1$ such that $Th_1 = h_2$.*

Except in trivial cases Problem 2.1.1 is ill-posed. Therefore we can repeat the words which have been written in connection with these problems in [T2]. At the same time, the use of bases with double orthogonality gives a more satisfactory approach to Problem 2.1.1 We describe this.

We denote by Π the operator of the orthogonal projection on Σ_1 in H_1, and by M the operator T^*T in H_1, where $T^* : H_2 \to H_1$ is the mapping adjoint to the mapping T according to the theory of Hilbert spaces.

PROPOSITION 2.1.2. *The restriction of the mapping ΠM to Σ_1 is a bounded linear operator from Σ_1 to Σ_1.*

PROOF. The norm of the operator ΠM is not greater than $m = \|T\|^2$ even in H_2. $\square$

PROPOSITION 2.1.3. *The operator $\Pi M : \Sigma_1 \to \Sigma_1$ is self-adjoint.*

PROOF. The restriction to Σ_1 of the operator ΠM coincides with the restriction to this space of the (evidently) self-adjoint operator $\Pi M \Pi$. $\square$

PROPOSITION 2.1.4. *The spectrum of the operator $\Pi M : \Sigma_1 \to \Sigma_1$ belongs to the segment $[0; m]$.*

PROOF. By Propositions 2.1.2 and 2.1.3 we can conclude that the spectrum of the operator ΠM belongs to the segment $[-m; m]$. On the other hand, this operator is non-negative, because for $h \in \Sigma_1$ we have

$$(\Pi M h, h)_{H_1} = (Mh, h)_{H_1} = \|Th\|^2_{H_2} \geq 0.$$

This proves our statement. $\square$

Problem 2.1.1 is definite if and only if the restriction of the operator T on Σ_1 is injective. A corresponding conclusion follows for the operator ΠM.

PROPOSITION 2.1.5. *The mappings $\Pi M : \Sigma_1 \to \Sigma_1$ and $T : \Sigma_1 \to \Sigma_2$ are simultaneously injective or not injective.*

PROOF. It is sufficient to prove that the kernels of these operators coincide. However, for $h \in \Sigma_1$, $\Pi M h = 0$ if and only if $(Mh, g)_{H_1} = (Th, Tg)_{H_2} = 0$ for all $g \subset \Sigma_2$, that is, if and only if $Th = 0$. This proves the proposition. $\square$

We can apply now the spectral theory of self-adjoint operators (see Riesz and Sz.-Nagy [RS-N], s. 107). Namely, let E_λ $(-\infty < \lambda < \infty)$ be an orthogonal decomposition of the unit in the Hilbert space Σ_1 corresponding to the operator ΠM. In the simplest case of a discrete spectrum $\lambda_1, \lambda_2, ...$ we have $E_\lambda = \sum_{\lambda \leq \lambda_j} pr_{\lambda_j}$ where pr_{λ_j} is the orthogonal projection to the eigen-subspace of ΠM corresponding to the eigenvalue λ_j. In the general case E_λ is some family of orthogonal projections concentrated on the spectrum of ΠM, and growing from 0 to I while λ changes from $-\infty$ to $+\infty$. This family has certain well known properties.

THEOREM 2.1.6 (ABSTRACT BERGMAN'S THEOREM). *Problem 2.1.1 is solvable if and only if*

$$\int_{-0}^{m} \frac{1}{\lambda^2} d(E_\lambda \Pi T^* h_2, \Pi T^* h_2)_{H_1} < \infty. \tag{2.1.1}$$

PROOF. The condition (2.1.1) means that the vector $\Pi T^* h_2 \in \Sigma_1$ belongs to the domain of the (left) inverse operator of the operator $\Pi M : \Sigma_1 \to \Sigma_1$. Hence one can find an element $h_1 \in \Sigma_1$ such that $\Pi M h_1 = \Pi T^* h_2$. This implies that the vector $M h_1 - T^* h_2 = T^*(T h_1 - h_2)$ is orthogonal to the subspace Σ_1 in H_1. In other words we have $(T^*(T h_1 - h_2), g)_{H_1} = (T h_1 - h_2), T g)_{H_2} = 0$ for all $g \in \Sigma_1$. Under the hypothesis, the vector h_2 belongs to the closure of the image of the mapping $T : \Sigma_1 \to \Sigma_2$. This means that one can find a sequence $\{f_j\} \subset \Sigma_2$ such that $T f_j$ converges to h_2 in H_2. Hence

$$\|T h_1 - h_2\|^2_{H_2} = \lim_{j \to \infty} (T h_1 - h_2, T(h_1 - f_j))_{H_2} = lim_{j \to \infty} 0 = 0,$$

therefore $T h_1 = h_2$. Thus, we see that the equalities $\Pi M h_1 = \Pi T^* h_2$ and $T h_1 = h_2$ are equivalent. This completes the proof of the theorem. □

From the proof of Theorem 2.1.6 one can see a curious phenomenon. Namely, if Problem 2.1.1 is solvable then its solution is unique. The formula for this solution is given in the following theorem.

THEOREM 2.1.7 (ABSTRACT CARLEMAN'S FORMULA). *Under condition (2.1.1) a solution of Problem 2.1.1 is given by the formula*

$$h_1 = \int_{-0}^{m} \frac{1}{\lambda} d(E_\lambda \Pi T^* h_2). \tag{2.1.2}$$

PROOF. Condition (2.1.1) guarantees the convergence of integral (2.1.2) in the weak topology of the space Σ_1. Therefore $h_1 \in \Sigma_1$ and we need only prove that $\Pi M h_1 = \Pi T^* h_2$. Now

$$\Pi M h_1 = \int_{0}^{m} \lambda \frac{1}{\lambda} d(E_\lambda \Pi T^* h_2) = \int_{-0}^{m} d(E_\lambda \Pi T^* h_2) = \Pi T^* h_2,$$

which was to be proved. □

We emphasize once again that under condition (2.1.1) the integral in formula (2.1.2) converges in the weak topology of the space Σ_1.

If we use the representation of the projections E_λ ($-\infty < \lambda < \infty$) by means of the eigenvectors of the operator $\Pi M : \Sigma_1 \to \Sigma_1$ (see Berezanskii [Bz]. ch. V) then we can see that it is possible to make formulae (2.1.1) and (2.1.2) more visible. For let $L_1 \subset \Sigma_1 \subset L_1'$ where L_1 is a topological vector space such that the embedding $L_1 \subset \Sigma_1$ is quasi-kernel, and the operator ΠM admits an extension $\Pi M : L_1 \to L_1$. Having taken the transposed mapping to this mapping we obtain a continuation of ΠM to a continuous linear operator on L_1' which is denoted by $\widetilde{\Pi M}$. Under the above assumption on L_1, the operator $\widetilde{\Pi M}$ has a complete system of generalized eigenvectors $\{b_\lambda^{(i)}\}_{\lambda\in\mathbb{R}}^{1\le i\le n_\lambda}$ in L_1' (see Berezanskii [Bz], p.341). This means that $\widetilde{\Pi M} b_\lambda^{(i)} = \lambda b_\lambda^{(i)}$, and for any vectors $h, g \in L_1$ there is Parseval's equality

$$(E(\Delta)h, g)_{H_1} = \int_\Delta \sum_{i=1}^{n_\lambda} (h, b_\lambda^{(i)})_{H_1} \overline{(g, b_\lambda^{(i)})_{H_1}} d\sigma(\lambda).$$

Here $E(\Delta) = \int_\Delta dE_\lambda$ is the spectral measure corresponding to the operator ΠM, and $d\sigma(\lambda)$ is a nonnegative Borel measure on the real axis. Using Parseval's equality for vectors in L_1 one can extend the "Fourier transformation" $(h, b_\lambda^{(i)})_{H_1}$ to vectors from Σ_1 by continuity. Then we have (in the sense of the $*$-weak convergence of the integrals in L_1')

$$E_\lambda h = \int_{-\infty}^{\lambda} \sum_{i=1}^{n_\lambda} (h, b_\zeta^{(i)})_{H_1} b_\zeta^{(i)} d\sigma(\zeta) \ \ (h \in \Sigma_1). \tag{2.1.3}$$

Corollary 2.1.8 (abstract Bergman's theorem). *Problem 2.1.1 is solvable if and only if*

$$\int_{-0}^{m} \sum_{i=1}^{n_\lambda} \left| \frac{(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}}{\lambda} \right|^2 d\sigma(\lambda) < \infty. \tag{2.1.4}$$

Proof. Using the equality (2.1.3), we obtain

$$d(E_\lambda \Pi T^* h_2, \Pi T^* h_2) = d \int_{-\infty}^{\lambda} \sum_{i=1}^{n_\zeta} |(\Pi T^* h_2, b_\zeta^{(i)})_{H_1}|^2 d\sigma(\zeta) =$$

$$\sum_{i=1}^{n_\lambda} |(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}|^2 d\sigma(\lambda).$$

In view of Theorem 2.1.6, we obtain the statement of the corollary. □

Corollary 2.1.9 (abstract Carleman's formula). *Under condition (2.1.1) a solution of Problem 2.1.1 is given by the following formula (where convergence is understood in the $*$-weak topology of the space L_1') :*

$$h_1 = \int_{-0}^{m} \sum_{i=1}^{n_\lambda} b_\lambda^{(i)} \frac{(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}}{\lambda} d\sigma(\lambda). \tag{2.1.5}$$

Proof. It is sufficient to calculate

$$dE_\lambda(\Pi T^* h_2) = \sum_{i=1}^{n_\lambda} b_\lambda^{(i)} (\Pi T^* h_2, b_\lambda^{(i)})_{H_1} d\sigma(\lambda).$$

and to put it in formula (2.1.2). □

We consider an instructive example.

EXAMPLE 2.1.10. We suppose that the operator $T : \Sigma_1 \to \Sigma_1$ is 1) injective, 2) compact. Then, by Proposition 2.1.4 the operator $\Pi M : \Sigma_1 \to \Sigma_1$ is injective, and (the compactness of T and) the boundedness of ΠT^* implies that $\Pi M : \Sigma_1 \to \Sigma_1$ is compact. According to the spectral theorem for compact self-adjoint operators (see Riesz and Sz.-Nagy [RS-N], s. 93) ΠM has in Σ_1 a countable complete system of eigenvectors $\{b_j\}_{j=1}^{\infty}$ corresponding to positive eigenvalues $\{\lambda_j\}$. However simple calculations show that $(Tb_j, Tb_j)_{H_2} = \lambda_j(b_j, b_j)_{H_1}$, that is, the system $\{Tb_j\}$ is orthogonal in Σ_2. Evidently this system is complete in Σ_1, hence it gives an orthogonal basis in this space. We notice that the system $\{b_j\} \subset \Sigma_1$ possesses the double orthogonality property : 1) relative to the scalar product $(.,.)_{H_1}$ in Σ_1 and 2) relative to the scalar product $(T.,T.)_{H_2}$ in Σ_1. As we noted in the introduction, Bergman was the first to devise these systems (see [Brg]), and Krasichkov [Kra] proved the abstract existence theorem. The orthogonal decomposition of the unit corresponding to the operator $\Pi M : \Sigma_1 \to \Sigma_1$ is now given by the operators $E_\lambda h = \sum_{\lambda \leq \lambda_j} b_j(h, b_j)_{H_1}$ (see (2.1.3)). Relations (2.1.4) and (2.1.5) take the form $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ and $h_1 = \sum_{j=1}^{\infty} c_j b_j$ respectively, where $c_j = \frac{(h_2, Tb_j)_{H_2}}{\|Tb_j\|^2_{H_2}}$ are Fourier coefficients of the vector $h \in \Sigma_2$ relative to the orthogonal basis $\{Tb_j\}$ in this space. $\square$

In the general case the system $\{b_\lambda^{(i)}\}$ also keeps some properties of bases with double orthogonality. We describe now an alternative method for its construction, using this idea. In the following we shall not take enough care of the legality of operations, because we want to make clear the idea only. The problem is first to construct a basis in Σ_2 and then to obtain by means of it a basis in Σ_1. We consider the operator $T\Pi T^* : \Sigma_2 \to \Sigma_2$. Again we notice that it is a bounded self-adjoint operator with the same spectrum, as ΠM. This operator is always injective, and it inherits the compactness property from $T : \Sigma_1 \to \Sigma_2$. We notice that the mapping $\Pi T^* : \Sigma_2 \to \Sigma_1$ is adjoint to $T : \Sigma_1 \to \Sigma_2$ in the sense of Hilbert spaces. To describe the image of T one can use an orthogonal decomposition of the unit $\{I_\lambda\}$ in Σ_2 corresponding to the operator $T\Pi T^*$. Then the solvability condition for Problem 2.1.1 has the form $\int_{-0}^{m} \frac{1}{\lambda} d(I_\lambda h_2, h_2) < \infty$, and the solution is given by the formula $h_1 = \Pi T^* \int_{-0}^{m} dI_\lambda(h_2)$. Further, the projection operators I_λ can be presented, similarly to (2.1.3), by generalized eigenvectors of the operator $T\Pi T^*$ in L_2', where $L_2 \subset \Sigma_2 \subset L_2'$ is a suitable equipment of the Hilbert space Σ_2. Let $\{e_\lambda^{(i)}\}$ be a complete system of these vectors in L_2'. Then, if the operator T is injective, $\{b_\lambda^{(i)}\}$ (where $b_\lambda^{(i)} = \frac{1}{\lambda}\Pi T^* e_\lambda^{(i)}$) is a complete system of generalized eigenvectors of the operator ΠM. We leave the reader to write the formulae, similar to (2.1.4) and (2.1.5), in terms of the system $\{e_\lambda^{(i)}\}$.

EXAMPLE 2.1.11. Krein and Nudelman [KrNu] have considered the Cauchy problem for holomorphic functions of the Hardy class H^2 in the lower half-plane with Cauchy data on the segment $[-1;1]$ of the real axis. They had $H_1 = L^2(\mathbb{R}^1)$, $H_2 = L^2([-1;1])$, the Hardy space Σ_1, and the operator of restriction $T : \Sigma_1 \to H_2$. In this case we have $\Sigma_2 = H_2$. The projection $\Pi : H_1 \to \Sigma_1$ is given by means of limit values on $\mathbb{R}^1$ of the Cauchy type integral in the lower half-plane. The operator $T\Pi T^* : \Sigma_2 \to \Sigma_2$ is an integral operator (but it is not the Carleman operator) with a simple spectrum. The complete system of generalized eigenfunctions of this operator was earlier constructed by Koppelman and Pincus [KpPi]. Having extrapolated it by the operator ΠT^* on the whole real axis, Krein and Nudelman [KrNu] obtained a continuous system of functions with double orthogonality in Σ_1. They also indicated a solvability condition, and a formula for the solutions of the Cauchy problem. $\square$

We finish this section with one more example connected with the Cauchy problem for holomorphic functions when the support of the Cauchy data is a "thin" set.

Example 2.1.12. Let σ be a compact set of positive measure in $\mathbb{R}^n$. We denote by W_σ the set of Fourier transforms of functions from $L^2(\sigma)$, that is, the set of functions of the type $\hat{u}(\zeta) = \frac{1}{(2\pi)^n} \int_\sigma e^{i\zeta x} u(x) dx$, where $u \in L^2(\sigma)$. According to the theorem of Paley and Wiener, elements of W_σ are restrictions on $\mathbb{R}^n$ of (not all!) entire functions of exponential order of growth in $\mathbb{C}^n$. For this reason W_σ is called the Wiener class. By means of the Plancheral theorem it is easy to see that W_σ is a closed subset of $L^2(\mathbb{R}^n)$. Let $S \subset \mathbb{R}^n$ be a given bounded set with a non-negative Borel measure m. In order not to complicate the notation we use the symbol $L^2(S)$ for the space of (classes of) functions which are measurable and square-integrable relative to the measure m on S. As for the assumptions about (S, m), we require that restrictions to S of (infinitely) differentiable functions in $\mathbb{R}^n$ should be contained in $L^2(S)$, and dense in this space. We consider the following problem: for a given function $u_0 \in L^2(S)$, find a function $u \in W_\sigma$ such that $u_{|S} = u_0$. To include it in the general scheme of Problem 2.1.1 we set $H_1 = \Sigma_1 = W_\sigma$, $H_2 = L^2(S)$, and define the operator $T : H_1 \to H_2$ as the restriction of functions on S. One can show that the operator T has a dense image. For let Φ be a continuous linear functional on $L^2(S)$ which vanishes on the image of T. According to the Riesz theorem, there is a function $\varphi \in L^2(S)$ such that $\Phi(u) = \int_S u\varphi dm$ for all $u \in L^2(S)$. Then one can consider Φ in explicit form as a distribution with compact support in $\mathbb{R}^n$. The condition $\Phi_{|imT} = 0$ implies that the Fourier transform $\hat{\Phi}$ of the distribution Φ vanishes on σ. Since $\hat{\Phi}$ is an entire function, and the measure of σ is positive then $\hat{\Phi} \equiv 0$ everywhere in $\mathbb{R}^n$. From this we conclude that Φ is the zero distribution in $\mathbb{R}^n$, that is, the zero functional on $L^2(S)$. Hence in our case we have $\Sigma_2 = H_2$. It is not difficult to verify that the operator T is compact. We shall assume its injectivity, in order that the Cauchy problem be defined. This simply means that S is a set of uniqueness for the class W_σ. Then we have the situation considered in Example 2.1.10. According to our earlier conclusions, if we denote by $\{b_j\}$, $j = 1, 2, ...$, a complete orthonormal system of eigenvectors of the operator T^*T in W_σ then the systems $\{Tb_j\}$, $j = 1, 2, ...$, will be an orthogonal basis in $L^2(S)$. The condition of solvability and the formula for solutions of the Cauchy problem have the forms $\sum_{j=1}^\infty |c_j|^2 < \infty$ and $u = \sum_{j=1}^\infty c_j b_j$ respectively, where $c_j = \frac{(u_0, Tb_j)_{L^2(S)}}{\|b_j\|^2_{L^2(S)}}$ are Fourier coefficients of the function u with respect to the orthogonal system $\{Tb_j\}$ in $L^2(S)$. If S is a set of positive measure in $\mathbb{R}^n$, then the results of this example were obtained by Krasichkov [Kra]. $\square$

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§2.2. The Cauchy problem for solutions of elliptic systems

We suppose that $D \Subset X$ is a domain with smooth boundary.

We fix a sufficiently small neighbourhood U of the boundary ∂D (as in §1.3) and a Dirichlet system of order $(p-1)$ on ∂D, say, $B_j \in do_{b_j}(E \to F_j)$ $(0 \le j \le p-1)$

where $F_j = U \times C^k$ are (trivial) bundles in U and using it we reformulate Problem 2.0 in the following form.

PROBLEM 2.2.1. *Let u_j $(0 \le j \le p-1)$ be sections of the bundles F_j over the set S. It is required to find a solution $u \in S_P(D)$ such that the expressions $B_j u$ $(0 \le j \le p-1)$ have in a suitable sense limit values on S coinciding with u.*

In order to justify the term "the Cauchy problem" for Problem 2.2.1, we note that the values of $B_j u$ $(0 \le j \le p-1)$ on S determine all the derivatives of u up to order $p-1$ on S. At the same time Problem 2.2.1 is solvable in the class of smooth (vector-) functions u (see Proposition 1.1.5), that is, it is not necessary to think about formal agreements between the sections u_j $(0 \le j \le p-1)$.

We denote by $S^f_P(D)$ the subspace of $S_P(D)$ which consists of solutions of finite order of growth near the boundary of D (see §1.3). As we have proved in §1.3, for any Dirichlet system of order $(p-1)$ on ∂D, say, $\{B_j\}$, we have $S^f_P(D) = S_{P,B}(D)$. For several reasons, it is convenient to consider the Cauchy Problem 2.2.1 in a subspace of $S^f_P(D)$. We indicate now a class of boundary sets S for which Problem 2.2.1 has no more than one solution in $S^f_P(D)$.

THEOREM 2.2.2. *Suppose that for a solution $u \in S^f_P(D)$ the boundary values $B_j u$ $(0 \le j \le p-1)$ vanish on a set $S \subset \partial D$ which has at least one interior point. Then $u \equiv 0$ in D.*

PROOF. Denote, as above, by $\mathcal{G}(\oplus B_j u)$ the integral on the left hand side of formula (1.3.1). Let $x^0 \in S$, and $B = B(x^0, r)$ be an open ball in X such that $B \cap \partial D \subset S$. We set $O = D \cup B$. Then $\mathcal{G}(\oplus B_j u) \in C^\infty_{loc}(E_{|O})$ satisfies $P\mathcal{G}(\oplus B_j u) = 0$ in the domain $O \subset X$, and it vanishes on the non-empty open subset $B \backslash D$ of this domain. Since the uniqueness property of the Cauchy problem in the small on X holds for P then $\mathcal{G}(\oplus B_j u) = 0$ in O. In particularly, $u \equiv 0$ in D, which was to be proved. $\square$

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§2.3. A solvability criterion of the Cauchy problem for elliptic systems in terms of surface bases with double orthogonality

In [T2] the maximal subclasses of $S^f(D)$ of solutions u, for which one can speak of the boundary values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D belonging to the range of usual (not generalized) sections of F_j, was distinguished. These are the so-called Hardy spaces $H^2_{P,B}(D)$ $(1 < q < \infty)$ which are modelled on the pattern of the classical Hardy spaces of holomorphic functions. One could say that $H^2_{P,B}(D)$ consists of all solutions $u \in S_{P,B}(D)$ for which the weak limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D belong to $L^2(F_{j|\partial D})$. In particular, with the topology induced by $L^2(\oplus F_{j|\partial D})$ the space $H^2_{P,B}(D)$ is a Hilbert space (see below). In this section we indicate an application of the abstract theory of §2.1 to the Cauchy Problem 2.2.1 in the Hardy class $H^2_{P,B}(D)$. So, let P be a (determined) elliptic differential operator whose transposed operator (P') satisfies the uniqueness condition for the Cauchy problem in the small on X. We consider the following problem.

Problem 2.3.1. *Let $u_j \in L^2(F_{j|S})$ $(0 \le j \le p-1)$ be known sections on S. It is required to find a solution $u \in H^2_{P,B}(D)$, satisfying $B_j u = u_j$ $(0 \le j \le p-1)$ on S.*

As was noticed by M.M. Lavrent'ev, the fundamental result about the solvability of Problem 2.3.1 is the following.

Lemma 2.3.2. *If the complement of S on ∂D has at least one interior point then Problem 2.3.1 is densely solvable.*

Proof. We denote by H the vector space $L^2(\oplus F_{j|S})$. Having provided each of the bundles F_j with some Hermitian metric $(.,.)_x$ we can define the conjugate linear isomorphism $* : F_j \to F_j^*$ by $< *\varphi, u >_x = (u, \varphi)_x$. With the scalar product $(\oplus u_j, \oplus \varphi_j)_H = \sum_{j=0}^{p-1} \int_S (u_j, \varphi_j)_x ds$ the vector space H is a Hilbert space. We consider in H the subset H_0 which is formed by elements of the form $\oplus B_j u$ where $u \in S_P(\overline{D})$. We obtain more than is asserted in the lemma if we prove that H_0 is dense in H. Using the Hahn-Banach theorem it is sufficient to show that if Φ is a continuous linear functional on H which is equal to zero on H_0, $\Phi \equiv 0$. Let Φ be such a functional. According to the theorem of Riesz, there are elements $\widetilde{\varphi}_j \in L^2(F_{j|S})$ $(0 \le j \le p-1)$ such that $\Phi(\oplus u_j) = (\oplus u_j, \oplus \widetilde{\varphi}_j)$ for all $\oplus u_j \in H$. Having extended each of the sections $\widetilde{\varphi}_j$ by zero to $\partial D \backslash S$ we obtain the sections $\varphi \in L^2(F_{j|\partial D})$ $(0 \le j \le p-1)$, and we set $g_j = *\varphi_j$, that is, $g_j \in L^2(F^*_{j|\partial D})$. Since the functional Φ vanishes on H_0, we have $\int_{\partial D} \sum_{j=0}^{p-1} < g_j, B_j u >_x ds = 0$ for all $u \in S(\overline{D})$. We can now use Theorem 29.3 from the book of Tarkhanov [T4] and conclude that there exists a section $g \in H^2_{P',C}(D)$ for which $C_j g = g_j$ $(0 \le j \le p-1)$ on ∂D. In particular, $C_j g = 0$ $(0 \le j \le p-1)$ on $\partial D \backslash S$. According to Theorem 2.2.2, $g \equiv 0$ in D, so that $\Phi \equiv 0$, which was to be proved. □

To apply the results of §2.1 to Problem 2.3.1 some information about the orthogonal projection in $L^2(\oplus F_{j|\partial D})$ on the subspace formed by elements of the form $\oplus B_j u$, where $u \in H^2_{P,B}(D)$, is needed. We can obtain it by the very general theory of functional spaces with reproducing kernels (see Aronszajn [Ar]). We now explain this. We consider the space $H^2_{P,B}(D)$ together with the Hermitian form

$$(u, v) = \sum_{j=0}^{p-1} \int_{\partial D} (B_j u, B_j \varphi)_x ds \quad (u, \varphi \in H^2_{P,B}(D)) \tag{2.3.1}$$

on it. Theorem 2.2.2 implies that any solution $u \in H^2_{P,B}(D)$ is completely defined by the restrictions of the expressions $B_j u$ $(0 \le j \le p-1)$ to ∂D. Hence the form (2.3.1) defines a scalar product on $H^2_{P,B}(D)$.

Lemma 2.3.3. *$H^2_{P,B}(D)$ is a separable Hilbert space.*

Proof. We can identify the pre-Hilbert space $H^2_{P,B}(D)$ with the subspace of $L^2(\oplus F_{j|\partial D})$ formed by the elements of the form $\oplus B_j u$, where $u \in H^2_{P,B}(D)$. However by Theorem 29.3 of see Tarkhanov [T4] one can quite simply notice that this subspace is closed. In fact, it is the intersection of kernels of special continuous linear functionals on $L^2(\oplus F_{j|\partial D})$. Hence, $H^2_{P,B}(D)$ inherits the properties of a closed subset of the separable Hilbert space. This proves the the lemma. □

Let x be a fixed point of the domain D. We consider the functional $\delta_x^{(j)}$ $(1 \leq j \leq k)$ on $H^2_{P,B}(D)$ given by $\delta_x^{(j)} f = f^{(j)}(x)$ $(1 \leq j \leq k)$ where $u^{(j)}(x)$ is the j-th component of u at the point x. Formula (1.3.1) implies that this functional is continuous on $H^2_{P,B}(D)$. Moreover, a stronger property than continuity holds. Namely, for any compact $K \subset D$ there is a constant C_K such that $\|\delta_x^{(j)}\| < C_K$ for $x \in K$. Hence, $H^2_{P,B}(D)$ is a space with a reproducing kernel (see Aronszajn [Ar]). We can now use the Riesz theorem on the general form of a continuous linear functional on a Hilbert space and thus find (unique) elements $\mathcal{K}_x^{(j)} \in H^2_{P,B}(D)$ $(1 \leq j \leq k)$ such that $u^{(j)}(x) = (u, \mathcal{K}_x^{(j)})_H$ for all $u \in H$. We denote by $\mathcal{K}_x^{(i,j)}$ $(1 \leq j, i \leq k)$ the i-th component of the vector-valued function $\mathcal{K}_x^{(j)}$. The (well defined) matrix $\mathcal{K}(x, y) = \|\mathcal{K}_x^{(i,j)}(y)\|$ is called the reproducing kernel of the domain D relative to $H^2_{P,B}(D)$. Its properties are well-known.

Proposition 2.3.4. *The matrix $\mathcal{K}(x, y)$ is Hermitian, that is, $\mathcal{K}(x, y)^* = \mathcal{K}(y, x)$.*

Proof. If $1 \leq j, i \leq k$ then

$$\mathcal{K}_y^{(i,j)}(x) = (\mathcal{K}_y^{(j)}, \mathcal{K}_x^{(i)})_H = \overline{(\mathcal{K}_x^{(i)}, \mathcal{K}_y^{(j)})_H} = \overline{\mathcal{K}_x^{(i,j)}(y)},$$

which was to be proved. $\square$

Proposition 2.3.5. $tr\mathcal{K}(x, x) = \sum_{j=1}^k \|\delta_x^{(j)}\|$.

Proof. We have,

$$tr\mathcal{K}(x, x) = \sum_{j=1}^k (\mathcal{K}_x^{(j)}, \mathcal{K}_x^{(j)})_H = \sum_{j=1}^k \|\delta_x^{(j)}\|,$$

which was to be proved. $\square$

Proposition 2.3.6. *If $\{e_\nu\}$ is an orthonormal basis of the space $H^2_{P,B}(D)$ then for all $x \in D$ we have $\mathcal{K}_x^{(j)} = \sum_{\nu=1}^\infty \overline{e_\nu^{(j)}(x)} e_\nu$ $(1 \leq j \leq k)$ where the series converges in the norm of $H^2_{P,B}(D)$. As a series of (vector-) functions of two variables $(x, y) \in D \times D$, it converges uniformly on compact subsets of $D \times D$.*

Proof. For a fixed $x \in D$ the Fourier series of the element $\mathcal{K}_x^{(j)} \in H^2_{P,B}(D)$ $(1 \leq j \leq k)$ with respect to the basis $\{e_\nu\}$ has the form $\mathcal{K}_x^{(j)} = \sum_{\nu=1}^\infty (\mathcal{K}_x^{(j)}, e_\nu)_H e_\nu$. To prove the first part of the proposition we notice that $(\mathcal{K}_x^{(j)}, e_\nu)_H = \overline{e_\nu^{(j)}(x)}$. We suppose now that K_i $(i = 1, 2)$ are compact subsets of D, and that constants C_i $(i = 1, 2)$ are chosen so that $\|\delta_x^{(j)}\| \leq C_i$ for $x \in K_i$. Then for $x \in K_i$

$$\left(\sum_{\nu=1}^\infty |\overline{e_\nu^{(j)}(x)}|^2 \right)^2 \leq \left| \sum_{\nu=1}^\infty \overline{e_\nu^{(j)}(x)} e_\nu(x) \right|^2 \leq$$

$$\leq C_i \left\| \sum_{\nu=1}^\infty \overline{e_\nu^{(j)}(x)} e_\nu(y) \right\|^2 = C_i \sum_{\nu=1}^\infty |e_\nu^{(j)}(x)|^2.$$

Hence here we have $\sum_{\nu=1}^{\infty} |e_\nu^{(j)}(x)|^2 \le C_i$ for $x \in K_i$ $(i = 1, 2)$. Thus, if $(x, y) \in K_1 \times K_2$, we obtain

$$\sum_{\nu=1}^{\infty} |\overline{e_\nu^{(j)}(x)} e_\nu(y)| \le \left(\sum_{\nu=1}^{\infty} |e_\nu^{(j)}(x)|^2 \right)^{1/2} \left(\sum_{\nu=1}^{\infty} |e_\nu(y)|^2 \right)^{1/2} \le \sqrt{k C_1 C_2}.$$

This proves the absolute and uniform convergence on compact subsets of $D \times D$ of the series for $\mathcal{K}_x^{(j)}$, which was to be proved. $\square$

The formula for the reproducing kernel mentioned in Proposition 2.3.6 could be written in the form $\mathcal{K}(x, y) = \sum_{\nu=1}^{\infty} e_\nu(x)^* \otimes e_\nu(y)$. The à priori estimates for a solution of an elliptic system imply that this series here converges uniformly together with all its derivatives on compact subsets of $D \times D$, that is, $\mathcal{K}$ is an infinitely differentiable section of $E \boxtimes E$ over $D \times D$.

THEOREM 2.3.7. *For all solutions $u \in H^2_{P,B}(D)$ the following formula holds*

$$u(x) = \int_{\partial D} \sum_{j=0}^{p-1} < *B_j \mathcal{K}(x, .), B_j u >_y ds \quad (x \in D). \tag{2.3.2}$$

PROOF. We simply rewrite the reproducing property of the kernel $\mathcal{K}$ in detail. $\square$

For holomorphic functions of several variables Theorem 2.3.7 is due to Bungart [Bu].

COROLLARY 2.3.8. *In the space $L^2(\oplus F_{j|\partial D})$ the operator of the orthogonal projection on the subspace Σ_1 formed by elements of the form $\oplus B_j u$ where $u \in H^2_{P,B}(D)$, has the form*

$$\Pi(\oplus u_j) = \oplus B_j \left(\int_{\partial D} \sum_{i=0}^{p-1} < *B_i \mathcal{K}(x, .), f_i >_y ds \right) \quad (\oplus u_j \in L^2(\oplus F_{j|\partial D}). \tag{2.3.3}$$

PROOF. Let $\{e_\nu\}$ be an orthonormal basis of the space $H^2_{P,B}(D)$. Then, from equality (2.3.1), $\{\oplus B_j e_\nu\}$ is an orthonormal basis of the subspace Σ_1 in $L^2(\oplus F_{j|\partial D})$. Hence if $\oplus u_j \in L^2(\oplus F_{j|\partial D})$ then

$$\Pi(\oplus u_j) = \sum_{\nu=1}^{\infty} (\oplus u_j, \oplus B_j e_\nu)_{L^2(\oplus F_{j|\partial D})} (\oplus B_j e_\nu) =$$

$$= \oplus B_j \left(\sum_{\nu=1}^{\infty} (\oplus u_j(y), \oplus B_j(y)(e_\nu^*(x) \otimes e_\nu(y)))_{L^2(\oplus F_{j|\partial D})} (\oplus B_j e_\nu) \right).$$

The first part of Proposition 2.3.6 implies that the sign of summation over ν can be taken inside sign of the scalar product. This gives at once formula (2.3.3), which was to be proved. $\square$

We outline a scheme of application of the theory of §2.1 to the Cauchy Problem 2.3.1. We set $H_1 = L^2(\oplus F_{j|\partial D})$ and $H_2 = L^2(\oplus F_{j|S})$. The Hermitian structures on these spaces are introduced as was explained in the proof of Lemma 2.3.2. Then H_1 and H_2 are Hilbert spaces. The operator $T : H_1 \to H_2$ is given by the restrictions of sections. Then the adjoint operator T^* is simply the extension of sections from S to $\partial D \backslash S$ by zero. Further, we consider in H_1 the subspace Σ_1 formed by elements of the form $\oplus B_j u$ where $u \in H^2_{P,B}(D)$. We have already noted that Σ_1 is a closed subspace of H_1 representing $H^2_{P,B}(D)$. We denote by Π the operator of orthogonal projection on Σ_1 in H_1. This is the integral operator given by formula (2.3.3). Lemma 2.3.2 means that the operator $T : \Sigma_1 \to H_2$ has a dense image, therefore we set $\Sigma_2 = H_2$. We must consider the mapping $\Pi T^* T : \Sigma_1 \to \Sigma_1$, which is given by the integral (2.3.3) except that the domain of integration is S instead of ∂D. If the set S has at least one interior point (on ∂D) then, from Theorem 2.2.2, the operators $T : \Sigma_1 \to \Sigma_2$ and $\Pi T^* T : \Sigma_1 \to \Sigma_1$ are injective. Even in the simplest situations the operator $\Pi T^* T$ is not compact, moreover, it is not Carleman operator (see Berezanskii [Bz], ch.V, 14). Let $\{b^{(i)}_\lambda\}$ be a complete system of generalized eigen vectors of the operator $\Pi T^* T$ in L_1' where $L \subset \Sigma_1 \subset L_1'$ is a suitable equipment of Σ_1. Then Corollaries 2.1.8 and 2.1.9 imply the following results.

Theorem 2.3.9. *We assume that the complement of S in ∂D has at least one interior point. Then for the solvability of Problem 2.3.1 it is necessary and sufficient that*

$$u(x) = \int_{-0}^{1} \sum_{i=1}^{N_\lambda} \left| \frac{(\Pi T^*(\oplus u_j), b^{(i)}_\lambda)_{H_1}}{\lambda} \right|^2 d\sigma(\lambda) < \infty \tag{2.3.4}$$

Proof. It is sufficient to note that in this case we have $m = \|T\|^2 = 1$. □

It is clear that Theorem 2.3.9 has only theoretical value, but is not in the least a practical, because its application depends on the singular eigenvalue problem for the operator $\Pi T^* T$. Therefore cases where one succeeds in calculating the system $\{b^{(i)}_\lambda\}$ in an explicit form are very interesting. There is such a situation in one of the simplest Cauchy problems for holomorphic functions, considered by Krein and Nudelman [KrNu] (see Example 2.1.11). A corresponding result holds for Carleman's formula.

Theorem 2.3.10. *Let $\partial D \backslash S$ have a non-empty interior (in ∂D). Then under condition (2.3.4) the solution of Problem 2.3.1 is given by the formula*

$$u(x) = -\int_{-0}^{1} \left(*^{-1} \sum_{i=1}^{N_\lambda} (\oplus C_j \mathcal{L}(x, .)), b^{(i)}_\lambda \right)_{H_1} \frac{(\Pi T^*(\oplus u_j), b^{(i)}_\lambda)_{H_1}}{\lambda} d\sigma(\lambda) \tag{2.3.5}$$

Proof. It is sufficient to substitute the expressions $\oplus B_j u(y)$ $(y \in D)$, obtained by Corollary 2.1.9, in Green's formula (1.3.1). □

A similar formula could be constructed on the basis of the integral representation (2.3.2). æ

§2.4. Green's integral and solvability of the Cauchy problem for (determined) elliptic systems

In this and the following 2 sections we assume that P is an elliptic differential operator such that the transposed operator P' satisfies the uniqueness condition of the Cauchy problem in the small on X.

Theorem 1.4.4 explains that if we solve Problem 2.2.1 (of Cauchy) in the class $S_P(D) \cap L^q(E_{|D})$ (or, more generally, in the class of sections satisfying $Pu = 0$ in D which have finite order of growth near the boundary of D) then we can hope only for generalized limit values of the expressions $B_j u$ $(0 \le j \le p-1)$ on ∂D. Therefore, since distributions have restrictions only on open subsets of the domain, it is natural to assume that S is an open connected piece (subdomain) of the boundary of D.

This situation can be realized in the following way. There is some domain $O \Subset X$, and S is a smooth closed hypersurface in O dividing this domain into two connected components: $O^- = D$ and $O^+ = O \backslash \overline{D}$.

In the wording of the following problem there are Besov spaces $B^{s-b_j-1/q,q}(F_{j|\overline{S}})$ whose definition may be not clear. We define these spaces in the following way. In Besov space $B^{s-b_j-1/q,q}(F_{j|\partial D})$ (defined by one of the usual method) we consider the subspace Σ formed by all the sections which are equal to zero on $\overline{S}$. For $s < 0$ this means that $< g, f >= 0$ for all $g \in B^{-s,q'}(F^*_{j|\partial D})$ with $supp\, g \subset \overline{S}$. It is easy to see that Σ is closed. The corresponding quotient space (with the quotient topology) we denote by $B^{s-b_j-1/q,q}(F_{j|\overline{S}})$

PROBLEM 2.4.1. *Let $u_j \in B^{s-b_j-1/q,q}(F_{j|\overline{S}})$ $(0 \le j \le p-1)$ be known sections on S where $s \in \mathbb{Z}_+$, and $1 < q < \infty$. It is required to find a section $u \in S_P(D) \cap W^{s,q}(E_{|D})$ such that $B_j u = u_j$ $(0 \le j \le p-1)$ on S.*

Under the formulated conditions the operator P has a right fundamental solution on X. In other words there is an operator $\mathcal{L} \in pdo_{-p}(F \to E)$ such that $\mathcal{L}P = 1 - \mathcal{S}^0$ on $C^\infty_\circ(E)$ where $\mathcal{S}^0 \in pdo_{-\infty}(E \to E)$ is some smoothing operator. Then $P\mathcal{S}^0 = 0$ on generalized sections of E with compact supports (that is, on $\mathcal{E}'(E)$).

Using the "initial" data of Problem 2.4.1 we construct Green's integral in a the special way. That is, we denote by $\widetilde{u}_j \in B^{s-b_j-1/q,q}(F_{j|\partial D})$ $(0 \le j \le p-1)$ an extension of the section u_j to the whole boundary. If, for example, $s = 0$ and $u_j \in L^2(F_{j|S})$ $(0 \le j \le p-1)$, then it is possible to extend them by zero on $\partial D \backslash S$. In any case the extensions could be chosen so that they will be supported on a given neighbourhood of the compact $\overline{S}$ on ∂D. Then we set $\widetilde{u} = \oplus \widetilde{u}_j$, and

$$\mathcal{G}(\widetilde{u})(x) = -\int_{\partial D} < C_j \mathcal{L}(x,.), \widetilde{u}_j >_y ds \quad (x \notin \partial D) \tag{2.4.1}$$

It is clear that $\mathcal{G}(\widetilde{u})$ is a solution of the system $Pu = 0$ everywhere in $X \backslash \partial D$. In particular, if we denote by $\mathcal{F}^\pm$ the restrictions of a section $\mathcal{F} \in \mathcal{D}'(E_{|O})$ to the sets $O^\pm$, then $\mathcal{G}(\widetilde{u})^\pm \in S_P(O^\pm)$.

THEOREM 2.4.2. *If the boundary of the domain D is sufficiently smooth then, for Problem 2.4.1 to be solvable, it is necessary and sufficient that the integral $\mathcal{G}(\widetilde{u})$ extends from O^+ to the whole domain O as a solution belonging to $S_P(O) \cap W^{s,q}(E_{|O})$.* ∎

PROOF. Necessity. Suppose that there is a section $u \in S_P(D) \cap W^{s,q}(E_{|D})$ such that $B_j u = u_j$ $(0 \le j \le p-1)$ on S.

We consider the following section in the domain O (more exactly, in $O \backslash S$):

$$\mathcal{F}(x) = \begin{cases} \mathcal{G}\widetilde{u}(x), x \in O^+, \\ \mathcal{G}\widetilde{u}(x) - u(x), x \in O^-. \end{cases} \tag{2.4.2}$$

Using the boundedness theorem for potential operators in Sobolev spaces on manifolds with boundary (see Rempel and Schulze [ReSz], 2.3.2.5) we can conclude that $\mathcal{G}(\widetilde{u})^{\pm} \in W^{s,q}(E_{|O^{\pm}})$ (here we need that $\partial D \in C^r$ with $r = max(s, p-s)$). This means $\mathcal{F}^{\pm} \in W^{s,q}(E_{|O^{\pm}})$.

On the other hand, we consider the difference $\delta = \mathcal{G}(\widetilde{u}) - \mathcal{G}(\oplus B_j u)$. Let $\varphi_\varepsilon \in \mathcal{D}(X)$ be any function supported on the ε-neighbourhood of the set $\partial D \backslash S$, and equal to 1 in some smaller neighbourhood of this set. Since $B_j u = \widetilde{u}_j$ $(0 \le j \le p-1)$ on S then we can write

$$\delta(x) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathcal{L}(x, .), \varphi_\varepsilon (B_j u - \widetilde{u}_j) >_y ds \quad (x \notin \partial D).$$

The right hand side of this equality is a solution of the system $Pu = 0$ everywhere in the domain O except the part of the ε-neighbourhood of the boundary of S on ∂D which belongs to O. Therefore, since $\varepsilon > 0$ is arbitrary, $\delta \in S_P(O)$.

Now using the expression for the integral $\mathcal{G}(\oplus B_j u)$ from Green's formula (1.3.4) and puting $\mathcal{G}(\widetilde{u}) = \mathcal{G}(\oplus B_j \widetilde{u}) + \delta$ in inequality (2.4.2) we obtain

$$\mathcal{F}(x) = \delta(x) \quad (x \in O \backslash S)$$

Since $\mathcal{S}^0(\chi_D u) \in S_P(X)$ the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $Pu = 0$.

Hence the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $Pu = 0$.

Thus, $\mathcal{F}$ belongs to $S_P(O) \cap W^{s,q}(E_{|O})$, and on O^+ this section coincides with $\mathcal{G}(\widetilde{u})^+$, which was to be proved.

Sufficiency. Conversely, let $\mathcal{F} \in S_P(O) \cap W^{s,q}(E_{|O})$ be a solution coinciding with $\mathcal{G}(\widetilde{u})^+$ on O^+. We set $u(x) = \mathcal{G}(\widetilde{u}) - \mathcal{F}(x)$ $(x \in D)$. The above mentioned boundedness theorem for potential operators in Sobolev spaces (see Rempel and Schulze [ReSz], 2.3.2.5) implies that $\mathcal{G}(\widetilde{u}) \in W^{s,q}(E_{|O^-})$. Therefore $u \in S_P(D) \cap W^{s,q}(E_{|D})$.

Now, for $g_j \in \mathcal{D}(F^*_{j|S})$ $(0 \le j \le p-1)$, Lemma 1.3.7 implies that

$$\lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j u(x - \varepsilon\nu(x)) >_x ds = \lim_{\varepsilon \to +0} \int_S < g, B_j u(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, B_j (\mathcal{G}(\widetilde{u}))(x - \varepsilon\nu(x)) - B_j \mathcal{F}(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, B_j (\mathcal{G}(\widetilde{u}))(x - \varepsilon\nu(x)) - B_j \mathcal{F}(x + \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, B_j (\mathcal{G}(\widetilde{u}))(x - \varepsilon\nu(x)) - B_j (\mathcal{G}(\widetilde{u}))(x + \varepsilon\nu(x)) >_x ds =$$

$$= \int_S < g_j, \widetilde{u}_j >_x ds = \int_S < g_j, u_j >_x ds.$$

Hence $B_j u = u_j$ $(0 \le j \le p-1)$ on S, that is, u is a soution of Problem 2.4.1, which was to be proved. □

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§2.5. A solvability criterion for the Cauchy problem for (determined) elliptic systems in the language of space bases with double orthogonality

Theorem 2.4.2 has been formulated so that the application of the theory of §2.1 is suggested. For this assume in addition that $q = 2$.

So, in this section we consider the solvability aspect of Problem 2.4.1.

PROBLEM 2.5.1. *Under what conditions on the sections* $u_j \in W^{s-b_j-1/2,2}(F_{j|\overline{S}})$ $(0 \le j \le p-1)$ *is there a solution* $u \in S_P(D) \cap W^{s,2}(E_{|D})$ *such that* $B_j u = u_j$ $(0 \le j \le p-1)$ *on* S *?*

Let Ω be some relatively compact subdomain of O^+. Since $\Omega \Subset O^+$, it follows that the restriction to Ω of Green's integral $\mathcal{G}(\widetilde{u})$ defined by equality (2.4.1) belongs to the space $S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$. Hence the extendibility condition for $\mathcal{G}(\widetilde{u})$ from O^+ to the whole domain O (as a solution in the class $S_P(O) \cap W^{s,2}(E_{|O})$) could be obtained by the use of a suitable system $\{b_\nu\}$ in $S_P(O) \cap W^{s,2}(E_{|O})$ with the double orthogonality property. More exactly, it is required that $\{b_\nu\}$ should be an orthonormal basis in $\Sigma_1 = S_P(O) \cap W^{s,2}(E_{|O})$ and an orthogonal basis in $\Sigma_2 = S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$ (or the contrary !).

How can such a system be constructed ? The theory of §2.1 answers this question.

We consider Sobolev spaces $H_1 = W^{s,2}(E_{|O})$ and $H_2 = W^{s,2}(E_{|\Omega})$ of sections of E. According to our approach we define them in the "interior" way using the Riemannian metric dx on O or Ω, and the Hermitian metric on (fibers of) E. Thus, H_1 and H_2 are Hilbert spaces. On the other hand, if the boundaries of O and Ω satisfy minimal conditions of the smoothness (roughly speaking they should be Lipschitz's ones) then these spaces are isomorphic (as normed spaces) to the Hilbert spaces $W^{s,2}(E_{|\overline{O}})$ and $W^{s,2}(E_{|\overline{\Omega}})$. These spaces are already defined in the "exterior" way. Namely, they are defined as quotient spaces of the Hilbert space $W^{s,2}(E)$ by closed subspaces of sections vanishing on $\overline{O}$ or $\overline{\Omega}$ respectively.

The operator $T : H_1 \to H_2$ is given by restriction of sections so that this is a continuous linear mapping of the Hilbert spaces.

Further, we distinguish in H_1 and H_2 the subspaces Σ_1 and Σ_2 which are formed by sections $\mathcal{F}$ satisfying $P\mathcal{F} = 0$ in O or Ω respectively. The Stiltjes-Vitali theorem (see Hörmander [Hö2], 4.4.2) implies that these subspaces are closed, therefore they are Hilbert spaces with the induced hermitian structures.

It is clear that the restriction of the map T to Σ_1 maps to Σ_2. However it is not evident that the image of T is dense in Σ_2.

LEMMA 2.5.2. *If the boundary of the domain* $\Omega \Subset O$ *is regular, and the complement of* Ω *has no compact connected components in* O *then the operator* $T : \Sigma_1 \to \Sigma_2$ *has a dense image.*

PROOF. We need to prove that restrictions to Ω of elements of $S_P(O) \cap W^{s,2}(E_{|O})$ are dense in $S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$ in the norm of $W^{s,2}(E_{|\Omega})$. However, since the boundary of Ω is regular, $S_P(\overline{\Omega})$ is dense in $S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$ in the norm of $W^{s,2}(E_{|\Omega})$ (see Tarkhanov [T4], ch. 4). On the other hand, the complement of Ω has no compact connected components in O, and hence the theorem of Runge implies that $S_P(\overline{O})$ is dense in $S_P(\overline{\Omega})$ (see the same book, theorem 11.26). Since $S_P(\overline{O}) \subset S_P(O) \cap W^{s,2}(E_{|O})$, and the natural topology in $S_P(\overline{O})$ is stronger than the induced topology from $W^{s,2}(E_{|O})$, we obtain the required result. $\square$

From the proof of the lemma we can see how to understand the words "regular boundary". If $s \geq p$, the word "regular" means any boundary. And if $s < p$ then this means that the complement of Ω in every boundary point is sufficiently massive. The reader can get a more exact characterization from the book of Tarkhanov [T4] (ch. 4).

LEMMA 2.5.3. *If the differential operator P satisfies the condition $(U)_S$ on X then the operator $T : \Sigma_1 \to \Sigma_2$ is injective*

PROOF. Let $u \in \Sigma_1$ and $Tf = 0$. This means that the solution $u \in S_P(O)$ vanishes on the non-empty open subset Ω of O. Hence the property $(U)_S$ implies $u \equiv 0$ everywhere in O, which was to be proved. $\square$

However the most important property of the operator T (in view of the application, via Theorem 2.4.2, of the theory of §2.1 to Problem 2.5.1) is the following.

LEMMA 2.5.4. *The operator $T : \Sigma_1 \to \Sigma_2$ is compact.*

PROOF. We need to show that the operator T maps any bounded set to a relatively compact set.

Let $K \subset \Sigma_1$ be a bounded set, that is, one can find a constant $C > 0$ such that $\|u\| < C$ for all $u \in K$. The image of K by the map T, that is, $T(K)$ is a relatively compact set if from any sequence $\{\mathcal{F}_j\} \subset T(K)$ one can extract a subsequence $\{\mathcal{F}_{jk}\}$ converging in Σ_2.

However if $\{\mathcal{F}_j\} \subset T(K)$ then $\mathcal{F}_j = u_{j|\Omega}$ where $\{u_j\} \subset K$. The sequence $\{u_j\}$ is bounded in the Hilbert space Σ_1. Therefore it contains a subsequence $\{u_{jk}\}$ which converges weakly to some element $u \in \Sigma_1$ (see Riesz and Sz.-Nagy [RS-N], s.32). Certainly $\{u_j\}$ converges to u in the topology of the space $\mathcal{D}'(E_{|O})$.

We use now the Stiltjes-Vitaly theorem (see Hörmander [Hö], 4.4.2) to conclude that $\{f_{jk}\}$ converges to u in the topology of the space $C^\infty_{loc}(E_{|O})$. We set $\mathcal{F} = u_{|\Omega}$, and $\mathcal{F}_{jk} = u_{jk|\Omega}$ then $\mathcal{F} \in \Sigma_2$ and $\{\mathcal{F}_{jk}\}$ converges to $\mathcal{F}$ in Σ_2, which was to be proved. $\square$

We can formulate now the main result on existence of bases with double orthogonality.

THEOREM 2.5.5. *If $\Omega \Subset O$ is an open set with a regular boundary whose complement (in O) has no compact connected components in O then in the space $S_P(O) \cap W^{s,2}(E_{|O})$ there is an orthonormal basis $\{b_\nu\}_{\nu=1}^\infty$ whose restriction to Ω is an orthogonal basis in $S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$.*

PROOF. We construct this basis by a method which will allow to obtain additional information about the corresponding eigen-value problem.

Let Π be the operator of orthogonal projection on Σ_1 in H_1. The à priori interior estimates for solutions of elliptic systems imply that the space Σ_1 (and Σ_2) is a Hilbert space with a reproducing kernel (see Aronszajn [Ar]). Hence Π is an integral operator with a kernel $\mathcal{K}(x, y) \in C^\infty_{loc}(E \boxtimes E_{|(O\times O)})$.

If $\{e_\nu\}_{\nu=1}^\infty$ is an orthonormal basis of the space $S_P(O) \cap W^{s,2}(E_{|O})$ then for all $x \in O$ we have $\mathcal{K}(x, .) = \sum_{\nu=1}^\infty e_\nu(x) \otimes e_\nu(.)$, where the series converges in the norm of $W^{s,2}(E \otimes E_{|O})$. As a series of (matrix-valued) functions of two variables $(x, y) \in O \times O$, this series converges uniformly on compact subsets of $O \times O$.

Thus, $\Pi\mathcal{F} = (\mathcal{F}, \mathcal{K}(x,.))_{H_1}$ $(\mathcal{F} \in H_1)$. Now simple calculations show that the operator $\Pi T^*T : H_1 \to H_2$ is integral. Namely,

$$(\Pi T^*T)\mathcal{F} = \int_\Omega \sum_{|\alpha|\leq s} < *D^\alpha K(x,.), D^\alpha \mathcal{F} >_y dv \quad (\mathcal{F} \in H_1).$$

From Lemmata 2.5.2, 2.5.3 and 2.5.4, and the results of Example 2.1.10 the restriction of the operator ΠT^*T to Σ_1 is injective, compact, and self-adjoint operator in Σ_1. Hence, if we denote by $\{b_\nu\}$ the countable complete orthonormal system of eigen-vectors of the operator ΠT^*T on Σ_1 (corresponding to eigenvalues $\{\lambda_\nu\} \subset (0,1)$), $\{b_\nu\}$ is an orthonormal basis of the space Σ_1 and $\{Tb_\nu\}$ is an orthogonal basis in Σ_2.

Therefore $\{b_\nu\}$ is a system with the double orthogonality property, which was to be proved. □

For an element $\mathcal{F} \in \Sigma_1$ we shall denote by $c_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthonormal system $\{b_\nu\}$ in Σ_1, that is, $c_\nu(\mathcal{F}) = (\mathcal{F}, b_\nu)_{H_1}$. And for an element $\mathcal{F} \in \Sigma_2$ we shall denote by $k_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthogonal system $\{Tb_\nu\}$ in Σ_2, that is, $k_\nu(\mathcal{F}) = \frac{(\mathcal{F}, Tb_\nu)_{H_2}}{(Tb_\nu, Tb_\nu)_{H_2}}$. Then the principal property of bases with double orthogonality is the following.

Lemma 2.5.6. *For any element $\mathcal{F} \in \Sigma_1$ we have*

$$c_\nu(\mathcal{F}) = k_\nu(T\mathcal{F}) \quad (\nu = 1, 2, ...) \tag{2.5.1}$$

Proof. Using the calculations of Example 1.9 we obtain

$$c_\nu(\mathcal{F}) = (\mathcal{F}, \frac{1}{\lambda_\nu}(\Pi T^*T)b_\nu)_{H_1} = \frac{1}{\lambda_\nu}(T\mathcal{F}, Tb_\nu)_{H_2} = k_\nu(T\mathcal{F}),$$

which was to be proved. □

We formulate now the solvability condition for Problem 2.5.1. Let $\mathcal{G}\widetilde{u}$ be Green's integral (see (2.4.1) constructed from the "initial" data of the problem. As already we noted, the restriction of the section $\mathcal{G}\widetilde{u}$ to Ω belongs to the space Σ_2.

Lemma 2.5.7. *For $\nu = 1, 2, ...$*

$$k_\nu(\mathcal{G}\widetilde{u}) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\mathcal{L}(., y)), \widetilde{u}_j >_y ds. \tag{2.5.2}$$

Proof. This consists of direct calculations with the use of equality (2.4.1). □

In order to determine the coefficients $k_\nu(\mathcal{G}\widetilde{u})$ $(\nu = 1, 2, ...)$ it is not necessary to know the basis $\{Tb_\nu\}$ in Σ_2. It is sufficient only to know the coefficients of the decomposition of the fundamental matrix $(\mathcal{L}(., y)$ $(y \in \partial D)$ with respect to this series. The properties of the coefficients $k_\nu(\mathcal{L}(., y) \in C^\infty_{loc}(F^*_{|X\setminus\Omega})$ we shall discuss in §2.6.

THEOREM 2.5.8. *If the boundary of the domain D is sufficiently smooth then for the solvability of Problem 2.5.1 it is necessary and sufficient that*

$$\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}\widetilde{u})|^2 < \infty. \tag{2.5.3}$$

PROOF. Necessity. Suppose that Problem 2.5.1 is solvable. Then Theorem 2.4.2 implies that the solution $\mathcal{G}\widetilde{u}$ extends from O^+ to the whole domain O as a solution belonging $S_P(O)\cap W^{s,2}(E_{|O})$. Having denoted this extension by $\mathcal{F}$ we obtain $\mathcal{F} \in \Sigma_1$ and $T\mathcal{F} = \mathcal{G}\widetilde{u}$ on Ω. Therefore taking into the consideration formula (2.5.1), and using Bessel's inequality we obtain

$$\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}\widetilde{u})|^2 = \sum_{\nu=1}^{\infty} |k_\nu(T\mathcal{F})|^2 = \sum_{\nu=1}^{\infty} |c_\nu(\mathcal{F})|^2 = \|F\|^2_{H_1} < \infty$$

which was to be proved.

Sufficiency. Conversely, let condition (2.5.3) hold. Then the theorem of Riesz and Fisher implies that there exists an element $\mathcal{F} \in \Sigma_1$ such that $c_\nu(\mathcal{F}) = k_\nu(\mathcal{G}\widetilde{u})$ for $\nu = 1, 2, ...$ Applying the operator T to the series $\mathcal{F} = \sum_{\nu=1}^{\infty} c_\nu(\mathcal{F}) b_\nu$ which converges in the norm of H_1, and taking into the consideration that the system $\{Tb_\nu\}$ is a basis in Σ_2, we have

$$T\mathcal{F} = \sum_{\nu=1}^{\infty} c_\nu(\mathcal{F}) Tb_\nu = \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{u}) Tb_\nu = \mathcal{G}\widetilde{u} \quad on \quad \Omega.$$

Hence $\mathcal{F} \in S_P(O) \cap W^{s,2}(E_{|O})$, and the restrictions to Ω of the sections $\mathcal{F}$ and $\mathcal{G}\widetilde{u}$ coincide. Since the differential operator P satisfies the condition $(U)_S$ on X it follows that the solution $\mathcal{F}$ coincides with $\mathcal{G}\widetilde{u}$ everywhere in O. We conclude now (using Theorem 2.4.2) that Problem 2.5.1 is solvable, which was to be proved. □

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§2.6. Carleman's formula

In this section we consider the regularization aspect of Problem 2.4.1.

PROBLEM 2.6.1. *It is required to find a solution $u \in S_P(D) \cap W^{s,2}(E_{|D})$ using known values $B_j u \in W^{s-b_j-1/2,2}(F_{j|\overline{S}})$ $(0 \le j \le p-1)$ on S.*

It is easy to see from Corollary 1.8 that side by side with the solvability conditions for Problem 2.4.1 ($q = 2$) bases with double orthogonality give the possibility of obtaining a suitable formula (of Carleman) for the regularization of solutions. We shall illustrate this on example of Problem 2.6.1.

Let $\{b_\nu\}$ be the basis with double orthogonality, constructed in the previous section, in the space $(\Sigma_1 =) S_P(O) \cap W^{s,2}(E_{|O})$ such that the restriction of $\{b_\nu\}$ to Ω (that is, $\{Tb_\nu\}$) is an orthogonal basis of $(\Sigma_2 =) S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$.

As above, we denote by $\{k_\nu(\mathcal{L}(., y))\}$ the sequence of Fourier coefficients for the fundamental matrix $\mathcal{L}(., y)$ $(y \in \Omega)$ with respect to the system $\{Tb_\nu\}$.

LEMMA 2.6.2. *The sections $k_\nu(\mathcal{L}(.,y))$ $(\nu = 1, 2...)$ are continuous, together with their derivatives up to order $(p-s-1)$, on the whole set X.*

PROOF. Though the restrictions to Ω of the columns of the fundamental matrix $\mathcal{L}(.,y)$ (for $y \in \Omega$) do not belong to the space Σ_2, for all $y \in X$ they do belong to $W^{p-1,q}(E_{|\Omega})$ where $q < \frac{n}{n-1}$. Hence the scalar products

(2.6.1)
$$k_\nu(\mathcal{L}(.,y)) = \frac{(\mathcal{L}(.,y), Tb_\nu)_{\Sigma_2}}{Tb_\nu, Tb_\nu)_{\Sigma_2}} = \frac{1}{\lambda_\nu} \sum_{|\alpha| \le s|} \int_\Omega < *D^\alpha b_\nu, D^\alpha \mathcal{L}(.,y) >_x dv \quad (\nu = 1, 2...)$$

are defined for all $y \in X$. Since $b_\nu \in C^\infty_{loc}(E_{|O})$ we have $k_\nu(\mathcal{L}(.,y)) \in C^{p-s-1}_{loc}(F^*)$. And this was to be proved. □

Using formula (2.6.1) one can see that the sections $k_\nu(\mathcal{L}(.,y))$ $(\nu = 1, 2...)$ extend to the boundary of Ω from each side as infinitely differentiable sections (at least, if the boundary is smooth).

LEMMA 2.6.3. *For any number $\nu = 1, 2, ...$ we have $P' k_\nu(\mathcal{L}(.,y)) = 0$ everywhere in $X \backslash \overline{\Omega}$.*

PROOF. Since $P'\mathcal{L}' = 1$ on $\mathcal{E}'(E^*)$ then (2.6.1) implies that

$$P' k_\nu(\mathcal{L}(.,y)) = P'\mathcal{L}'(\chi_\Omega(*b_\nu)) = \chi_\Omega(*b_\nu) \quad (\nu = 1, 2, ...),$$

and this proves the statement. □

We introduce the following kernels $\mathfrak{C}^{(N)}$ defined for $(x, y) \in O \times X$ $(x \neq y)$:

$$\mathfrak{C}^{(N)}(x,y) = \mathcal{L}(x,y) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\mathcal{L}(.,y)) \quad (N = 1, 2, ...). \tag{2.6.2}$$

LEMMA 2.6.4. *For any number $N = 1, 2, ...$ the kernels $\mathfrak{C}^{(N)} \in C_{loc}(E \boxtimes F)$ satisfy $P(x)\mathfrak{C}^{(N)}(x,y) = 0$ for $x \in O$, and $P'(y)\mathfrak{C}^{(N)}(x,y) = 0$ for $y \in X \backslash \Omega$ everywhere except on the diagonal $\{x = y\}$.*

PROOF. Since $\{b_\nu\} \subset S_P(O)$, this immediately follows from Lemma 2.6.3. □

In the following lemma $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{b_\nu\}$.

LEMMA 2.6.5. *Let $h = h(\alpha)$ be a continuous map of a topological space $\mathcal{A}$ to $\mathcal{H}$. Then, for any element $h(\alpha)$, the Fourier series converges uniformly with respect to α on compact subsets of $\mathcal{A}$.*

PROOF. Let $(.,.)$ be the scalar product and $\|h\| = (h,h)^{1/2}$ be a norm in $\mathcal{H}$ $(h \in \mathcal{H})$.

We fix arbitrary $\alpha \in \mathcal{A}$ and denote by $c_\nu(\alpha)$ the Fourier coefficients of the vector $h(\alpha)$ with respect to the system $\{b_\nu\}$: $c_\nu(\alpha) = (h(\alpha), b_\nu)$. Then for any $\varepsilon > 0$ there is $N > 0$, $N = N(\varepsilon, \alpha)$, such that for every $m \ge N$ the following inequality holds:

$$\|h(\alpha) - \sum_{\nu=1}^{m} c_\nu(\alpha) b_\nu\| = \left(\|h(\alpha)\|^2 - \sum_{\nu=1}^{m} |c_\nu(\alpha)|^2 \right)^{1/2} \le \varepsilon. \tag{2.6.3}$$

Since the map h and the scalar product $(.,.)$ are continuous, there is a neighbourhood $\mathcal{V}_N(\alpha)$ of the point α in which estimate (2.6.3) still holds for $m = N$. However, if m increases, the right hand side of (2.6.3) can only decrease. Therefore inequality (2.6.3) holds in the neighbourhood $\mathcal{V}_N(\alpha)$ for all $m \geq N$.

Now, for any compact $K \subset \mathcal{A}$, we can choose $N_1 = N_1(K)$ such that estimate (2.6.3) holds for all $\alpha \in K$ because we can cover the compact by a finite number of neighbourhoods of the type $\mathcal{V}_N(\alpha)$. The proof is complete. $\square$

From the following lemma one can see that the sequence of kernels $\{\mathfrak{C}^{(N)}\}$ interpolated for real values $N \geq 0$ in a suitable way, for example in the piece-constant way, gives special Carleman's function for Problem 2.6.1 (see Tarkhanov [T4], §25).

LEMMA 2.6.6. *For any multi-index α, $D_y^\alpha \mathfrak{C}^{(N)}(.,y) \to 0$ in the norm of $W^{s,2}(E \otimes F^*_{y|O})$ uniformly with respect to y on compact subsets of $X \backslash \overline{O}$, and even $X \backslash O$ if $|\alpha| < p - s - n/2$.*

PROOF. First, we notice that, if $y \in X \backslash \overline{O}$, every column of the matrix $\mathcal{L}(x.,y)$ is an element of the space Σ_1. Therefore using Lemma 2.5.6 we obtain $\mathfrak{C}^{(N)}(.,y) = \mathcal{L}(.,y) - \sum_{\nu=1}^N c_\nu(\mathcal{L}(.,y))$. Differentiating this identity with respect to y we find the equality

$$D_y^\alpha \mathfrak{C}^{(N)}(.,y) = D_y^\alpha \mathcal{L}(.,y) - \sum_{\nu=1}^{N} b_\nu \otimes c_\nu(D_y^\alpha \mathcal{L}(.,y)) \quad (y \in X \backslash \overline{O}). \tag{2.6.4}$$

The correspondence $y \to D_y^\alpha \mathcal{L}(.,y)$ defines a continuous linear mapping of the topological space $X \backslash \overline{O}$ to the direct sum of k copies of the space Σ_1. Therefore for every column of the matrix $D_y^\alpha \mathcal{L}(.,y)$ its Fourier series with respect to the orthonormal basis $\{b_\nu\}$ converges in the norm of Σ_1 uniformly with respect to y on compact subsets of $X \backslash \overline{O}$ (see Lemma 2.6.5). This proves the first part of the lemma. As for the second part, it is sufficient to use the same arguments because for $|\alpha| < p - s - n/2$ the correspondence $y \to D_y^\alpha \mathcal{L}(.,y)$ defines a continuous linear mapping of the whole set $X \backslash \overline{O}$ to the direct sum of k copies of the space Σ_1. $\square$

We can formulate now the main result of the section. For $u \in S_P(D) \cap W^{s,2}(E_{|D}))$ we denote by $\widetilde{u} \in W^{s-b_j-1/2,2}(F_{j|\partial D})$ $(0 \leq j \leq p-1)$ some (arbitrary) extensions of the sections $B_j u$ from S to the whole boundary.

THEOREM 2.6.7 (CARLEMAN'S FORMULA). *For any solution $u \in S_P(D) \cap W^{s,2}(E_{|D})$ the following formula holds:*

$$u(x) = -\lim_{N \to \infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathfrak{C}^{(N)}(x,.), \widetilde{u}_j >_y ds \quad (x \in D). \tag{2.6.5}$$

PROOF. Let $\mathcal{G}(\widetilde{u})$ be Green's integral constructed by formula (2.4.1). Theorem 2.5.8 implies that $\sum_{\nu=1}^\infty |k_\nu(\mathcal{G}(\widetilde{u})| < \infty$. Hence, from the theorem of Riesz and Fisher, there exists an element $\mathcal{F} \in S_P(O) \cap W^{s,2}(E_{|O})$ such that $c_\nu(\mathcal{F}) = k_\nu(\mathcal{G}(\widetilde{u}))$. In proving Theorem 2.5.8 we saw that this solution $\mathcal{F}$ is an extension of $\mathcal{G}(\widetilde{u})$ from the domain O^+ to the whole domain O as a solution in $S_P(O) \cap W^{s,2}(E_{|O})$.

Then Theorem 2.4.2 implies that the section $u'(x) = \mathcal{G}(\widetilde{u})(x) - \mathcal{F}(x)$ $(x \in D)$ belongs to $S_P(D) \cap W^{s,2}(E_{|D})$, and satisfies $B_j u' = u$ $(0 \le j \le p-1)$ on S. Using (uniqueness) Theorem 2.2.2 we see that $u = u'$ everywhere in D. Hence

$$u(x) = (\mathcal{G}(\widetilde{u}))(x) - \mathcal{F}(x) = (\mathcal{G}(\widetilde{u}))(x) - \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu(x) =$$

$$= (\mathcal{G}(\widetilde{u}))(x) - \lim_{N\to\infty} \sum_{\nu=1}^{N} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu(x). \tag{2.6.6}$$

Puting in (2.6.6) the expressions for the coefficients $k_\nu(\mathcal{G}(\widetilde{u}))$ $(\nu = 1, 2, ...)$ which are given in Lemma 2.5.7 we obtain

$$u(x) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathcal{L}(x,.), \widetilde{u}_j >_y ds +$$

$$+ \lim_{N\to\infty} \left(\sum_{\nu=1}^{N} \int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\mathcal{L}(x,.)), \widetilde{u}_j >_y ds \right) b_\nu(x) =$$

$$= -\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \left(\mathcal{L}(x,.) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\mathcal{L}(x,.)) \right), \widetilde{u}_j >_y ds =$$

$$= -\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathfrak{C}^{(N)}(x,.), \widetilde{u}_j >_y ds,$$

which was to be proved. $\square$

We emphasize that the integral in the right hand side of formula (2.5.4) depends only on values of the expressions $B_j u$ $(0 \le j \le p-1)$ on S. Thus, this formula is a quantitative expression of (uniqueness) Theorem 2.2.2. However this gives much more than the uniqueness theorem because there is sufficiently complete information about Carleman's function $\mathfrak{C}^{(N)}$.

For harmonic functions Carleman' formula (2.6.5) is first met, apparently, in [Sh1].

Remark 2.6.8. The series $\sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{u})b_\nu$ (defining the solution $\mathcal{F}$) converges in the norm of the space $W^{s,2}(E_{|O})$. The Stiltjes-Vitali theorem (see Hörmander [Hö2], 4.4.2) implies now that it converges together with all its derivatives on compact subsets of O. Then, from formula (2.6.6), one can see that the limit in (2.6.5) is reached in the topology of the space $C^\infty_{loc}(E_{|O})$.

In fact, the opposite statement (for Theorem 2.6.7) holds. For the Cauchy-Riemann system this fact was proved by Aizenberg (see [AKSh]).

THEOREM 2.6.9. *Let for sections* $u_j \in W^{s-b_j-1/2,2}(F_{j|S})$ $(0 \le j \le p-1)$ *there exist in the norm of the space* $W^{s,2}(E_{|D)}$ *the limit*

$$v(x) = -\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathfrak{C}^{(N)}(x,.), \widetilde{u}_j >_y ds \quad (x \in D).$$

Then $v \in S_P(D) \cap W^{s,2}(E_{|D})$ *and* $B_j v_{|S} = u_j$*, i.e.* v *is the solution of Problem 2.6.1 for* $\oplus u_j$*.*

PROOF. (The Uniqueness) Theorem 2.2.2 implies that it suffices to prove that Problem 2.6.1 is solvable for the sections u_j $(0 \le j \le p-1)$. In order to prove this we prove that the series $\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}(\widetilde{u}))|^2$ with the Fourier coefficients $k_\nu(\mathcal{G}(\widetilde{u}))$ of the integral $\mathcal{G}(\widetilde{u})$ with respect to the orthogonal system $\{b_{\nu|\Omega}\}$ converges (see Theorem 2.5.8).

By the definition $\{b_\nu\}$ is orthonormal in $W^{s,2}(E_{|O})$; therefore

$$\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}(\widetilde{u}))|^2 = \left\| \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu \right\|^2_{W^{s,2}(E_{|O})} =$$

$$= \left\| \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu \right\|^2_{W^{s,2}(E_{|O-})} + \left\| \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu \right\|^2_{W^{s,2}(E_{|O+})}. \tag{2.6.7}$$

Let us prove, first, the boundedness of the first summand in (2.6.7). By simple calculations

$$v(x) = (\mathcal{G}(\widetilde{u}))(x) - \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu(x) \ (x \in D).$$

By the hypothesis of the theorem $v \in W^{s,2}(E_{|D})$; moreover $\mathcal{G}(\widetilde{u}) \in W^{s,2}(E_{|D})$ (see [ReSz], 2.3.2.5). Hence

$$\left\| \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu \right\|^2_{W^{s,2}(E_{|O-})} \le \|v\|^2_{W^{s,2}(E_{|O-})} + \|\mathcal{G}(\widetilde{u})\|^2_{W^{s,2}(E_{|O-})} < \infty.$$

To finish the proof we need to show that $\|\sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u}))b_\nu\|^2_{W^{s,2}(E_{|O+})} < \infty$. However, for any point $x \in O^+$ there is a domain Ω_x, with smooth boundary, such that $\Omega \Subset \Omega_x \Subset O^+$ and the complement of Ω_x in O has no compact connected components. Proving Lemma 2.5.2, we have seen that under conditions above the system $\{b_\nu\}$ is dense in $S(\Omega_x) \cap W^{s,2}(E_{|\Omega_x})$ (in the norm of the last space). Then, because $\mathcal{G}(\widetilde{u}) \in W^{s,2}(E_{|\Omega_x})$, the following decomposition holds:

$$\mathcal{G}(\widetilde{u})(x) = \sum_{\nu=1}^{\infty} a_\nu(\mathcal{G}(\widetilde{u}), \Omega_x)b_\nu(x) \ (x \in \Omega_x). \tag{2.6.8}$$

On the other hand, because $\{b_\nu\}$ is an orthogonal basis in $S(\Omega) \cap W^{s,2}(E_{|\Omega})$, we have

$$\mathcal{G}(\widetilde{u})(x) = \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u})) b_\nu(x) \ (x \in \Omega). \tag{2.6.9}$$

Comparing (2.6.8) and (2.6.9) we conclude that $k_\nu(\mathcal{G}(\widetilde{u})) = a_\nu(\mathcal{G}\widetilde{u}, \Omega_x)$ for every $\nu \in \mathbb{N}$. Hence decomposition (2.6.9) holds for $x \in O^+$.

Finally, results of [ReSz] (see 2.3.2.5) imply that $\mathcal{G}(\widetilde{u}) \in W^{s,2}(E_{|O^+})$. Therefore

$$\left\| \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}(\widetilde{u})) b_\nu \right\|^2_{W^{s,2}(E_{|O^+})} = \|\mathcal{G}(\widetilde{u})\|^2_{W^{s,2}(E_{|O^+})} < \infty.$$

The proof is complete. $\square$

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§2.7. A stability set in the Cauchy problem for elliptic systems

As we have already noted the Cauchy problem for elliptic systems is ill-posed (see, for example, [Hd]). In this section we consider the stability aspect of Problem 2.4.1. More exactly, we are aimed in finding a stability set in the problem. That means a set Σ of solutions $u \in W^{s,2}(E_{|D})$ to $Pu = 0$ such that $B_j u^{(\mu)} \to 0$ on S $(0 \le j \le p-1)$ implies $u^{(\mu)} \to 0$ in D, for any sequence $\{u^{(\mu)}\} \subset \Sigma$.

We consider Green's integral

$$\widetilde{\mathcal{G}}(\oplus u_j)(x) = -\int_S \sum_{j=0}^{p-1} < C_j \mathcal{L}(x, .), u_j > ds \ (x \notin S),$$

$\{C_j\}_{j=0}^{p-1}$ being the Dirichlet system on ∂D adjoint to $\{B_j\}_{j=0}^{p-1}$ with respect to Green's formula, (see Lemma 1.1.6) and $\mathcal{L}$ being a fundamental solution of P on X.

Let, as in §2.5, $\Omega \Subset O$ be a domain with regular boundary whose complement in O has no compact connected components. This integral, when restricted on Ω, is in $S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$. Denote by $k_\nu(\widetilde{\mathcal{G}}(\oplus u_j))$ its Fourier coefficients with respect to the orthogonal system $\{b_{\nu|\Omega}\}$. Exactly,

$$k_\nu(\widetilde{\mathcal{G}}(\oplus u_j)) = -\int_S \sum_{j=0}^{p-1} < C_j k_\nu(\mathcal{L}(., y)) u_j > ds \ (\nu \in \mathbb{N})$$

where $k_\nu(\mathcal{L}(., y))$ are the Fourier coefficients of the fundamental solution $\mathcal{L}(x, y)_{|x \in \Omega}$, y being on S.

We complete our results on the solvability of Cauchy the problem in the following way.

THEOREM 2.7.1. *Given a sequence* $\{u^{(\mu)}\} \subset S_P(D) \cap W^{s,2}(E_{|\Omega})$, *if for every* $\mu \in \mathbb{N}$

$$\sum_{\nu=1}^{\infty} |k_\nu(\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)}))|^2 \le 1$$

and $B_j u^{(\mu)} \to 0$ *in the norm of* $W^{s-b_j-1/2,2}(G_{j|S})$ *for all* $0 \le j \le p-1$, *then* $u^{(\mu)} \to 0$ *in the topology of* $W^{s,2}_{loc}(E_{|D\cup S})$.

Remark 2.7.2. Of course, the adequate conclusion here would be that $u^{(\mu)} \to 0$ in the norm of $W^{s,2}(E_{|D})$, but we are not able to prove that.

PROOF. Fix a sequence $\{u^{(\mu)}\}$ satisfying the condition above.

Arguing as in the proofs of Theorems 2.4.2 and 2.5.8 one obtains that

$$u^{(\mu)} = \widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)})(x) - \mathcal{F}_\mu(x) \ (x \in D) \tag{2.7.1}$$

where $\mathcal{F}_\mu \in S_P(O) \cap W^{s,2}(E_{|O})$ is given by the Fourier series

$$\mathcal{F}_\mu = \sum_{\nu=1}^{\infty} k_\nu(\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)})b_\nu.$$

We see at once that the restrictions of $\mathcal{F}_\mu$ and $\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)})$ to Ω coincide.

Since $B_j u^{(\mu)} \to 0$ in the norm of $W^{s-b_j-1/2,2}(G_{j|S})$ for all $0 \le j \le p-1$, the first term in the right hand side of (2.7.1) tends to zero in the topology of $W^{s,2}_{loc}(E_{|D\cup S})$.

We claim that $\mathcal{F}_\mu \to 0$ in the topology of $C^\infty_{loc}(E_{|O})$. To prove this it suffices to show that each subsequence of $\{u^{(\mu)}\}$ has a subsequence which converges to zero in $C^\infty_{loc}(E_{|O})$.

Indeed, assume that this is true while $\mathcal{F}_\mu$ does not converge to zero. As

$$\|\mathcal{F}_\mu\|^2_{W^{s,2}(E_{|O})} = \sum_{\nu=1}^{\infty} |k_\nu(\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)}))|^2 \le 1$$

and the embedding

$$S_P(O) \cap W^{s,2}(E_{|O}) \to \{v \in C^\infty_{loc}(E_{|O}) : Pv = 0 \ in \ O\}$$

is compact, each subsequence of $\{\mathcal{F}_\mu\}$ contains a subsequence convergent in the topology of $C^\infty_{loc}(E_{|O})$. Therefore, $\{\mathcal{F}_\mu\}$ has a sequence which converges to a nonzero element of $C^\infty_{loc}(E_{|O})$. This contradicts the our assumption.

We now turn to proving the relation $\mathcal{F}_\mu \to 0$ in $C^\infty_{loc}(E_{|O})$. To this end, we take a subsequence of $\{\mathcal{F}_\mu\}$, which we again denote by $\{\mathcal{F}_\mu\}$.

From discussion above it follows that $\{\mathcal{F}_\mu\}$ has a subsequence $\{\mathcal{F}_{\mu_i}\}$ which converges in the topology of $C^\infty_{loc}(E_{|O})$ to a function $\mathcal{F} \in C^\infty_{loc}(E_{|O})$ satisfying $P\mathcal{F} = 0$ in O. In particular, the sequence $\{\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)})_{|\Omega}\}$ converges to $\mathcal{F}_{|\Omega}$.

On the other hand, $\{\widetilde{\mathcal{G}}(\oplus B_j u^{(\mu)})_{|\Omega}\}$ converges to zero on Ω because the Cauchy data of $u^{(\mu)}$ on S tends to zero. Thus, $\mathcal{F} = 0$ on Ω, and so $\mathcal{F} \equiv 0$ in the domain O.

This completes the proof. □

Remark 2.7.3. One could formulate similar result in terms of the Fourier coefficients $k_\nu(\mathcal{G}(\oplus B_j \widetilde{u}^{(\mu)}))$.

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§2.8. Examples for the Laplace operator in $\mathbb{R}^n$

2.8.1 Solvability condition for the Cauchy problem for the Laplace operator in $\mathbb{R}^n$ in terms of Green's integral.

In this subsection we consider the following variant of the Cauchy problem 2.4.1.

Again let O be a bounded domain in $\mathbb{R}^n$ and S be a closed smooth hypersurface dividing it into 2 connected components: O^+ and $O^- = D$, and oriented as the boundary of O^-.

PROBLEM 2.8.1.1. *Under what conditions on functions $u_0 \in C^1(S)$ and $u_1 \in C^0(S)$ is there a function $u \in C^1(D \cup S)$, which is harmonic in D and such that the restrictions on S of u and its normal derivative $\frac{\partial u}{\partial n}$ are equal to u_0 and u_1 correspondingly.*

In other words, we consider the situation where $P = \Delta_n$ is the Laplace operator in $\mathbb{R}^n$, $B_0 = 1$ and $B_1 = \frac{\partial}{\partial n}$.

We denote by σ_n the area of the unit sphere in $\mathbb{R}^n$ and by $\varphi_n(y)$ the standard (bilateral) fundamental solution of the Laplace operator in $\mathbb{R}^n$:

$$\varphi_n(y) = \begin{cases} \frac{1}{(2-n)\sigma_n |y|^{n-2}}, & n > 2, \\ \frac{1}{2\pi} ln|y|, & n = 2. \end{cases}$$

Assume that the functions u_0, u_1 are summable on S. Then the corresponding Green's integral is well defined:

$$\mathcal{G}(\oplus u_j) = \int_S \left(u_0(y) \frac{\partial \varphi_n(x-y)}{\partial n_y} - u_1(y)\varphi_n(x-y) \right) ds(y) \ (x \in O \backslash S).$$

It is clear that $\mathcal{G}(\oplus u_j)$ is harmonic everywhere outside of S; let $\mathcal{G}(\oplus u_j)^{\pm} = \mathcal{G}(\oplus u_j)_{|O^{\pm}}$.

Theorem 2.4.2 and Lemma 1.3.4 imply the following result.

THEOREM 2.8.1.2. *Let $S \in C^2$, $u_0 \in C^1$ and $u_1 \in C^0$ be summable functions on S. Then, for Problem 2.8.1.1 to be solvable, it is necessary and sufficient that the integral $\mathcal{G}(\oplus u_j)^+$ harmonically extends from O^+ to the domain O.*

PROOF. See also paper [Sh1].

EXAMPLE 2.8.1.3. Let S be a piece of the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n$. Then, if $u_0 = 0$, the function $\mathcal{G}(\oplus u_j)$ is even with respect to $x_n \neq 0$, and, if $u_1 = 0$, it is odd. Therefore, if one of the functions u_j $(0 \leq j \leq 1)$ is zero, the integrals $\mathcal{G}(\oplus u_j)^{\pm}$ extend harmonically across S simultaneously. Because their difference on S is equal to u_0, and the difference of their normal derivatives is equal to u_1, Theorem 2.8.1.2 implies the known Hadamard's statement (see [Hd]. p. 31). Namely, if one of the functions u_j $(0 \leq j \leq 1)$ is zero, Problem 2.8.1.1 is solvable only if another function is real analytic.

2.8.2 Example of a basis with double orthogonality in the Cauchy problem for the Laplace operator in $\mathbb{R}^n$.

Let $O = B_R$ be the ball with centre at zero and radius $0 < R < \infty$, and S be a closed smooth hypersurface dividing it into 2 connected components (O^+ and $O^- = D$) in such a way that $0 \in O^+$, and oriented as the boundary of O^-. In this case we can construct a basis with double orthogonality in the subspace $S^{s,2}_{\Delta_n}(B_R)$ of $W^{s,2}(B_R)$ ($s \geq 0$), which consists of harmonic functions, in a rather explicit form.

For $s \in \mathbb{Z}_+$, we provide $W^{s,2}(B_R)$ with the scalar product

$$(u,v)_{W^{s,2}(B_R)} = \int_{|y|\leq R} \sum_{|\alpha|\leq s} (D^\alpha u)(y)\overline{(D^\alpha v)(y)}dy \quad (u,v \in W^{s,2}(B_R)).$$

Hence $S^{s,2}_{\Delta_n}(B_R)$ is a Hilbert space with the induced from $W^{s,2}(E_{|B_R})$ Hilbert structure.

According to Theorem 1.4.4 for $u \in S^{s,2}_{\Delta_n}(B_R)$ there exists weak boundary values $(D^\alpha u)_{|\partial B_R}$ belonging to the Sobolev space $W^{s-|\alpha|-1/2,2}(E_{|\partial B_R})$. Then, for $s = N - 1/2$ ($N \in \mathbb{N}$) we provide $S^{s,2}_{\Delta_n}(B_R)$ with the scalar product

$$(u,v)_{S^{s,2}_{\Delta_n}(B_R)} = (u,v)_{W^{[s],2}(\partial B_R)} = \int_{|y|=R} \sum_{|\alpha|\leq [s]} (D^\alpha u)(y)\overline{(D^\alpha v)(y)}d\sigma(y),$$

where $u,v \in S^{s,2}_{\Delta_n}(B_R)$ and $[s]$ is the integral part of s. It is not difficult to see that in this case $S^{s,2}_{\Delta_n}(B_R)$ is a Hilbert space too with the topology equivalent to the one induced from $W^{s,2}(B_R)$.

For example, $S^{1/2,2}_{\Delta_n}(B_R)$ is the Hardy space of harmonic complex valued functions in B_R, $S^{1/2,2}_{\Delta_n}(B_R) \subset S^{0,2}_{\Delta_n}(B_R)$ and these spaces are not equal (cf. [ShT1]).

For other non-integer s we will define a special Hilbert structure in $S^{s,2}_{\Delta_n}(B_R)$ later.

Let $\{h^{(i)}_\nu\}$ be a set of homogeneous harmonic polynomials which form a complete orthonormal system in $L^2(\partial B_1)$ (*spherical harmonics*) where ν is the degree of homogeneity, and i is an index labeling the polynomials of degree ν belonging to the basis. The size of the index set for i as a function of ν is known, namely, $1 \leq i \leq J(\nu)$ where $J(\nu) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ for $n > 2$ and $\nu \geq 0$ (see [So], p. 453). If $n = 2$ then, obviously, $J(0) = 1$, $J(\nu) = 2$ for $\nu \geq 1$. Using the system $\{h^{(i)}_\nu\}$ we will construct the basis with double orthogonality.

The following decomposition for $\varphi_n(x-y)$ can be found for even $n > 2$ in [AKy]) and for the general case in [Sh1] (Lemma 3.2).

Lemma 2.8.2.1.

$$\varphi_n(x-y) = \varphi_n(y) - \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h^{(i)}_\nu(x)}{n+2\nu-2}\frac{\overline{h^{(i)}_\nu(y)}}{|y|^{n+2\nu-2}}, \tag{2.8.2.1}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

Proof. Because of the homogeneity of the polynomial $h^{(i)}_\nu$, Euler formula implies that

$$\sum_{m=1}^{n} \frac{\partial h^{(i)}_\nu}{\partial x_m}x_m = \nu h^{(i)}_\nu, \quad \sum_{m=1}^{n} \frac{\partial^2 h^{(i)}_\nu}{\partial x_m \partial x_j}x_m = (\nu-1)\frac{\partial h^{(i)}_\nu}{\partial x_j}x_j. \tag{2.8.2.2}$$

We denote by $Y_\nu^{(i)}$ the restriction of the polynomial $h_\nu^{(i)}$ to ∂B_1. Then $\{Y_\nu^{(i)}\}$ is a basis in $L^2(\partial B_1)$ consisting of spherical functions.

Let $x \in B_1$ be fixed. We represent $\varphi_n(x-y)$ by the Fourier series in $L^2(\partial B_1)$. Namely,

$$\varphi_n(x-y) = \sum_{\nu,i} c_\nu^{(i)}(x)\overline{Y_\nu^{(i)}},$$

where $c_\nu^{(i)}(x)$ are the Fourier coefficients of $\varphi_n(x-y)$ with respect to the system $\{Y_\nu^{(i)}\}$.

Let us consider first the case where $n > 2$. Then

$$c_\nu^{(i)}(x) = \frac{1}{(2-n)\sigma_n}\int_{\partial B_1} |x-y|^{2-n} Y_\nu^{(i)}(y)d\sigma(y),$$

where $d\sigma$ is the volume form on the sphere ∂B_1. We rewrite the coefficients in the following way:

$$c_\nu^{(i)}(x) = \frac{1}{(2-n)}\int_{\partial B_1} \mathfrak{P}(x,y)\frac{1-2<x,y>+|x|^2}{1-|x|^2} Y_\nu^{(i)}(y)d\sigma(y). \tag{2.8.2.3}$$

Here $<x,y>=\sum_{m=1}^n x_m y_m$ and

$$\mathfrak{P}(x,y) = \frac{1}{\sigma_n}\frac{1-|x|^2}{|x-y|^n}$$

is the Poisson kernel for the unit ball in $\mathbb{R}^n$.

It is not difficult to see that the function

$$\mathcal{F}(x) = x_m h_\nu^{(i)}(x) - \frac{1}{n+2\nu-2}\frac{\partial h_\nu^{(i)}}{\partial x_m}(|x|^2-1) \tag{2.8.2.4}$$

is the harmonic extension into the ball B_1 of the function $y_m Y(i)_\nu$ given on ∂B_1. Really, using (2.8.2.2) and harmonicity of $h(i)_\nu$ we have:

$$\Delta_n \mathcal{F} = 2\frac{\partial h_\nu^{(i)}}{\partial x_m}(x) - \frac{1}{n+2\nu-2}\frac{\partial h_\nu^{(i)}}{\partial x_m}(x)\Delta_n(|x|^2-1)+$$

$$+\frac{2}{n+2\nu-2}\sum_{j=1}^n \frac{\partial^2 h_\nu^{(i)}}{\partial x_m \partial x_j}(x)\frac{\partial}{\partial x_j}(|x|^2-1) =$$

$$= 2\frac{\partial h_\nu^{(i)}}{\partial x_m}(x) - \frac{2}{n+2\nu-2}\left(n\frac{\partial h_\nu^{(i)}}{\partial x_m}(x) + 2\sum_{j=1}^n \frac{\partial^2 h_\nu^{(i)}}{\partial x_m\partial x_j}(x)x_j\right) = 0.$$

Using the Poisson formula and equalities (2.8.2.2), (2.8.2.3) and (2.8.2.4) we obtain

$$c_\nu^{(i)}(x) = \frac{1}{(2-n)}\frac{1+|x|^2}{1-|x|^2}\int_{\partial B_1}\mathfrak{P}(x,y)Y_\nu^{(i)}(y)d\sigma(y) -$$

$$-\frac{2}{(2-n)}\sum_{m=1}^n \frac{x_m}{1-|x|^2}\int_{\partial B_1}\mathfrak{P}(x,y)y_m Y_\nu^{(i)}(y)d\sigma(y) = -\frac{h_\nu^{(i)}(x)}{n+2\nu-2}.$$

Therefore

$$\varphi_n(x-y) = -\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)\overline{Y_\nu^{(i)}(y)}}{n+2\nu-2},$$

and Lemma 2.6.5 implies that this series converges in the norm of the space $L^2(\partial B_1)$, uniformly with respect to x on compact subsets of the ball B_1.

The harmonic extension with respect to y leads us to the equality

$$|y|^{2-n}\varphi_n(x-\frac{y}{|y|}) = -\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)\overline{h_\nu^{(i)}(y)}}{n+2\nu-2},$$

where the series converges absolutely and uniformly with respect to x and y inside the ball B_1.

Applying to this equality the Kelvin transformation with respect to y we see that

$$\varphi_n(x-y) = -\sum_{\nu=0}^{\infty}\sum_{i=0}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}. \tag{2.8.2.5}$$

It is clear that series (2.8.2.5) converges uniformly with respect to x (inside the ball B_1) and y (outside $\overline{B}_1$). Let us show that it is converges uniformly on the set of the following type

$$\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : \frac{|y|}{|x|}\geq\delta_1,\ and\ |y|\geq\delta_0\}$$

where $\delta_1>1$, $\delta_0>0$. We choose $\gamma>1$ such that $\gamma^2<\delta_1$. Then

$$\sum_{i=0}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} =$$

$$= \left(\frac{\gamma}{|y|}\right)^{n-2}\sum_{i=0}^{J(\nu)} \frac{h_\nu^{(i)}(\frac{x}{\gamma|x|})}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(\frac{y}{\gamma|y|})}\left(\frac{\gamma^2|x|}{|y|}\right)^{\nu}}{|y|^{n+2\nu-2}}.$$

By the choice of γ we have:

$$\left|\frac{x}{\gamma|x|}\right| = \frac{1}{\gamma}<1,\ \left|\frac{\gamma y}{|y|}\right| = \gamma>1,\ \frac{\gamma^2|x|}{|y|}\leq\frac{\gamma^2}{\delta_1}<1.$$

Using the criterion of Abel for the uniform convergence of series and the following estimate of a harmonic homogeneous polinomial h_ν of degree ν on the unit sphere (see [So]):

$$\max_{|y|=1}|h_\nu|\leq const(n)\nu^{n/2-1}\|h_\nu\|_{L^2(\partial B_1)}, \quad \square \tag{2.8.2.6}$$

we see that series (2.8.2.5) converges absolutely together with all its derivatives, uniformly on subsets of the type above.

If $\nu=0$ then $J(0)=1$ and $h_0^{(1)}=const$. Because the system $\{h_\nu^{(i)}\}$ is orthonormal we conclude that $|h_0^{(1)}|^2=\frac{1}{\sigma_n}$. Therefore

$$\varphi_n(x-y)=\frac{1}{(2-n)\sigma_n|y|^{n-2}}-\sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)}\frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.$$

In the case $n=2$, we have

$$c_\nu^{(i)}(x)=\frac{1}{2\pi}\int_{\partial B_1}Y_\nu^{(i)}(y)ln|x-y|d\sigma(y).$$

However, from the discussion above, we see that, for $\nu\geq 1$ and $m=1,2$,

$$\frac{\partial c_\nu^{(i)}}{\partial x_m}(x)=\frac{1}{2\pi}\int_{\partial B_1}\frac{x_m-y_m}{|y-x|^2}Y_\nu^{(i)}(y)d\sigma(y)=\frac{-1}{2\nu}\frac{\partial h_\nu^{(i)}}{\partial x_m}(x).$$

Moreover, because $\nu\geq 1$, $c_\nu^{(i)}(0)=h_\nu^{(i)}(0)=0$. Hence

$$c_\nu^{(i)}(x)=-\frac{h_\nu^{(i)}(x)}{2\nu}\ (\nu\geq 1)$$

If $\nu=0$ then

$$\frac{\partial c_1^{(1)}}{\partial x_m}(x)=\frac{h_0^{(1)}}{2\pi}\int_{\partial B_1}\frac{x_m-y_m}{|y-x|^2}Y_\nu^{(i)}(y)d\sigma(y)=$$

$$=\frac{h_0^{(1)}}{2\nu(1-|x|^2)}\left(x_m\int_{\partial B_1}\mathfrak{P}(x,y)d\sigma(y)-\int_{\partial B_1}y_m\mathfrak{P}(x,y)d\sigma(y)\right)=0\ (m=1,2).$$

Arguing as before we obtain:

$$\frac{1}{2\pi}ln|x-y|=\frac{1}{2\pi}ln|y|-\sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)}\frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.\qquad\square$$

Lemma 2.8.2.2. *The system $\{h_\nu^{(i)}\}$ is an orthogonal basis in $S^{s,2}_{\Delta_n}(B_R)$ ($s=(N-1)/2$, $N\in\mathbb{N}$). Moreover there exist constants $C_1(s,n)$, $C_2(s,n)>0$ such that*

$$C_1(s,n)\|h_\nu^{(i)}\|^2_{S^{s,2}_{\Delta_n}(B_R)}\leq\nu^{2s}\|h_\nu^{(i)}\|^2_{L^2(E_{|B_R})}\leq C_2(s,n)\|h_\nu^{(i)}\|^2_{S^{s,2}_{\Delta_n}(B_R)}$$

for every $\nu\geq 0$, $1\leq i\leq dimS_k(\nu)$.

Proof. Let us first check the orthogonality of the system $\{h_\nu^{(i)}\}$. Using the homogeneity of the polynomials, one easily obtains

$$\sum_{|\alpha|=m}\int_{|y|\leq R}(D^\alpha h_\nu^{(i)})^*(y)(D^\alpha h_\mu^{(j)})(y)dy=$$

$$= \begin{cases} 0, \ m > \nu \ or \ m > \mu \ or \ \mu \neq \nu; \\ \frac{R^{n+2\nu-2m-1}}{n+2\nu-2m-1} \int_{|y|=1} \sum_{|\alpha|=m} (D^\alpha h_\nu^{(i)})^*(y)(D^\alpha h_\nu^{(j)})(y) d\sigma(y), \end{cases}$$

which implies (with $m = 0$) the orthogonality for the case $s = 1/2$, and then

$$\sum_{|\alpha|=m} \int_{|y|\le R} (D^\alpha h_\nu^{(i)})^*(y)(D^\alpha h_\mu^{(j)})(y) dy =$$

$$= R^{n+\mu+\nu-2m+1} \sum_{|\beta|=m-1} \int_{|y|=1} (D^\beta h_\nu^{(i)})^*(y) \sum_{i=1}^{n} y_i \frac{(D^\beta h_\mu^{(j)})(y)}{\partial y_i} d\sigma(y) =$$

$$= \begin{cases} (\nu - m + 1) R^{n+2\nu-2m+1} \sum_{|\beta|=m-1} \int_{|y|=1} \sum_{i=1}^{n} (D^\beta h_\nu^{(i)})^* (D^\beta h_\nu^{(j)}) d\sigma(y), \ \nu \geq m, \\ 0, \ \nu \leq m-1, \ or \ \nu \neq \mu \end{cases}$$

which implies (with $m = 1$) the orthogonality for the case $s = 1$.

Arguing by induction, we obtain that the system $\{h_\nu^{(i)}\}$ is orthogonal in $S^{s,2}_{\Delta_n}(B_R)$.

The estimates follow immediately from the calculations above.

Let us prove that the system $\{h_\nu^{(i)}\}$ is dense in $S^{m,2}_{\Delta_n}(B)$.

It is known that a function $u \in S^{m,2}_{\Delta_n}(B)$ can be approximated in the norm of the space $W^{m,2}(B)$ by functions u_N ($N = 1, 2, \ldots$), which are harmonic in a neighbourhood of the ball $\overline{B}$ (see, for example, [T4], ch. 4). Because, for every ($N = 1, 2, \ldots$), the function u_N is harmonic in a neighbourhood of a (larger than B) ball $\hat{B}$, it can be represented in the ball $\hat{B}$ by Green's formula (1.1.2) with the fundamental solution $\mathcal{L}(x, y) = \varphi_n(x - y)$. Substituting in this Green's formula decomposition (2.8.2.1), we obtain a sequence $\{u_{NM}\}$ of finite linear combinations of polynomials $h_\nu^{(i)}$ which converges to u_N in the norm of $W^{m,2}(B)$. Taking the diagonal sequence $\{u_{NN}\}$ we obtain the desired approximation of u in the norm of $W^{m,2}(B)$. The proof is complete. $\square$

Now, for $s \geq 0$ ($s \neq (N-1)/2$, $N \in \mathbb{N}$) we provide the space $S^{s,2}_{\Delta_n}(B_R)$ with the Hermitian form

$$(u, v)_{S^{s,2}_{\Delta_n}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} C_\nu^{(i)}(u) \overline{C_\nu^{(i)}(v)} \nu^{2s} \ (u, v \in S^{s,2}_{\Delta_n}(B_R)),$$

where $C_\nu^{(i)}(u)$ are the Fourier coefficients of the vector-function u with respect to the orthonormal basis $\{h_\nu^{(i)}\}$ in $S^{0,2}_{\Delta_n}(B_R)$.

Proposition 2.8.2.3. *The Hermitian form* $(.,.)_{S^{s,2}_{\Delta_n}(B_R)}$ ($s \geq 0$) *is a scalar product in* $S^{s,2}_{\Delta_n}(B_R)$ *defining a topology, equivalent to the original one. Moreover, the system* $\{h_\nu^{(i)}\}$ *is an orthogonal basis in* $S^{s,2}_{\Delta_n}(B_R)$ *and there exist constants* $C_1(s, n)$, $C_2(s, n) > 0$ *such that*

$$C_1(s, n) \|h_\nu^{(i)}\|^2_{W^{s,2}(E_{|B_R})} \leq \nu^{2s} \|h_\nu^{(i)}\|^2_{L^2(B_R)} \leq C_2(s, n) \|h_\nu^{(i)}\|^2_{W^{s,2}(E_{|B_R})}$$

for every $\nu \geq 0$, $1 \leq i \leq dim S_k(\nu)$.

PROOF. For $s = (N-1)/2$, $N \in \mathbb{N}$ the statement was proved in Lemma 2.8.2.2.

We fix a number $s \geq 0$ ($s \neq (N-1)/2$, $N \in \mathbb{N}$) and consider 2 interpolation couples $S^{[s],2}_{\Delta_n}(B_R)$, $S^{[s]+1,2}_{\Delta_n}(B_R)$ and $l_2([s])$, $l_2([s]+1)$, where $[s]$ is the integral part of s and, for $r \geq 0$,

$$l_2(r) = \{\{K_\nu\}_{\nu=0}^{\infty} : \sum_{\nu=1}^{\infty} |K_\nu|^2 \nu^{2r} < \infty\}.$$

Then, see, for example, [Tr] (4.1 - 4.4 and 1.18.2), we have interpolation spaces with $0 < \gamma < 1$

$$[S^{[s],2}_{\Delta_n}(B_R), S^{[s],2}_{\Delta_n}(B_R)]_\gamma = S^{[s]+\gamma,2}_{\Delta_n}(B_R),$$
$$[l_2([s]), l_2([s]+1)]_\gamma = l_2([s]+\gamma).$$

Now, for $u \in S^{s,2}_{\Delta_n}(B_R)$ denote by Mu the sequence $\{\sum_{i=1}^{dim S_k(\nu)} |C^{(i)}_\nu(u)|^2\}_{\nu=0}^{\infty}$, where $C^{(i)}_\nu(u)$ are the Fourier coefficients of the vector-function u with respect to the orthonormal basis $\{h^{(i)}_\nu\}$ in $S^{0,2}_{\Delta_n}(B_R)$. According to Lemma 2.8.2.2, the operator

$$M : S^{m,2}_{\Delta_n}(B_R) \to l_2(m)$$

is continuous for every $m \in \mathbb{Z}_+$. Therefore, using standart interpolation arguments (see [Tr]), we conclude that the operator

$$M : S^{[s]+\gamma,2}_{\Delta_n}(B_R) \to l_2([s]+\gamma)$$

is continuous for every $0 < \gamma < 1$. In particular, $(.,.)_{S^{[s]+\gamma,2}_{\Delta_n}(B_R)}$ defines a weaker topology that the one induced from $W^{[s]+\gamma,2}(E_{|B_R})$.

The estimates follow from Lemma 2.8.2.2 and Interpolation Theory (see [Tr], 1.3.3, p. 25).

The system $\{h^{(i)}_\nu\}$ is complete in $S^{s,2}_{\Delta_n}(B_R)$ because it is complete in $S^{0,2}_{\Delta_n}(B_R)$ and orthogonal in $S^{s,2}_{\Delta_n}(B_R)$. $\square$

We fix $0 < r < dist(0,S)$ and set $\Omega = B_r$ so that $\Omega \Subset O$. In order to obtain the Fourier coefficients for the section $\mathcal{G}(\widetilde{u})$ with respect to this basis in $S^{0,2}_{\Delta_n}(B_r)$ it is sufficient to know the Fourier coefficients for the fundamental solution $\varphi_n(x-y)$ (see Lemma 2.8.2.1.).

Our principal results will be formulated in the language of the coefficients

$$k^{(i)}_\nu = \begin{cases} \frac{-1}{n+2\nu-2} \int_S \left(u_0(y) \frac{\partial}{\partial n} \left(\frac{\overline{h^{(i)}_\nu(y)}}{|y|^{n+2\nu-2}} \right) - u_1(y) \frac{\overline{h^{(i)}_\nu(y)}}{|y|^{n+2\nu-2}} \right) ds(y) \ (\nu = 1,2,...), \\ \int_S \left(u_0(y) \frac{\partial \varphi_n(y)}{\partial n} - u_1(y)\varphi_n(y) \right) ds(y), \nu = 0. \end{cases}$$

THEOREM 2.8.2.4. *Let u_0, $u_1 \in L^1(S)$. Then for Problem 2.8.1.1 to be solvable, it is necessary and sufficient that*

(2.8.2.7) $$\limsup_{\nu \to \infty} \max_i \sqrt[\nu]{|k^{(i)}_\nu(y)|} \leq \frac{1}{R}$$

PROOF. Necessity. Let Problem 2.8.1.1 be solvable. Then Theorem 2.8.1.2 implies that the function $\mathcal{G}\widetilde{u}^+$ on the domain O^+ harmonically extends to a function $\mathcal{F} \in S_{\Delta_n}(B_R)$.

We fix $0 < r < R$. It is clear that the components of the solution $\mathcal{F}$ belong to the space $S^{0,2}_{\Delta_n}(B_r)$. Therefore, from Lemma 2.8.2.2, they are represented by their Fourier series with respect to the orthonormal in $S^{0,2}_{\Delta_n}(B_r)$ system $\{\sqrt{\frac{n+2\nu}{r^{n+2\nu}}}h^{(i)}_\nu\}$

$$\mathcal{F}(x) = \sum_{i,\nu} c^{(i)}_\nu(r)\sqrt{\frac{n+2\nu}{r^{n+2\nu}}}h^{(i)}_\nu(x) \quad (x \in B_r). \tag{2.8.2.8}$$

Bessel's inequality implies that the series $\sum_{i,\nu}|c^{(i)}_\nu(r)|^2$ converges. On the other hand, in the ball Ω, from Lemma 2.8.2.1, we obtain the decomposition

$$\mathcal{G}(\oplus u_j)(x) = \sum_{i,\nu} k^{(i)}_\nu h^{(i)}_\nu(x) \quad (x \in \Omega). \tag{2.8.2.9}$$

Comparing (2.8.2.9) and (2.8.2.8) we find that

$$c^{(i)}_\nu(r) = \sqrt{\frac{r^{n+2\nu}}{n+2\nu}}k^{(i)}_\nu \quad (\nu = 1, 2, ...).$$

Hence for any $0 < r < R$

$$\sum_{i,\nu}|k^{(i)}_\nu(r)|^2\frac{r^{n+2\nu}}{n+2\nu} = r^n\sum_{\nu=0}^{\infty}\left(\sum_{i=1}^{J(\nu)}\frac{|k^{(i)}_\nu(r)|^2}{n+2\nu}\right)r^{2\nu} < \infty$$

Using the Cauchy-Hadamard formula for the radius of the convergence of a power series we obtain

$$\limsup_{\nu\to\infty}\max_i \sqrt[\nu]{|k^{(i)}_\nu(y)|} \leq \limsup_{\nu\to\infty}\left(\sum_{i=1}^{J(\nu)}\frac{|k^{(i)}_\nu(r)|^2}{n+2\nu}\right)^{1/2\nu} \leq \frac{1}{r}$$

Since $0 < r < R$ is arbitrary then condition (2.8.2.7) holds, which was to be proved.

Sufficiency. If condition (2.8.2.7) holds then the Cauchy-Hadamard formula and the estimate $J(\nu) < const\, \nu^{n-2}$ implies that the series $\sum_{i,\nu}|k^{(i)}_\nu(r)|^2\frac{r^{n+2\nu}}{n+2\nu}$ converges for any $0 < r < R$. The Riesz-Fisher theorem implies that there exists a section $\mathcal{F}$ (of the bundle $E_{|B_r}$) with the components from $S^{0,2}_{\Delta_n}(B_r)$ such that

$$\mathcal{F}(x) = \sum_{i,\nu}\sqrt{\frac{r^{n+2\nu}}{n+2\nu}}k^{(i)}_\nu\sqrt{\frac{n+2\nu}{r^{n+2\nu}}}h^{(i)}_\nu(x) =$$

$$= \sum_{i,\nu} k^{(i)}_\nu h^{(i)}_\nu(x)$$

where the series converges in the norm of the space $L^2(B_r)$. It is easy to see that in the ball Ω the section $\mathcal{F}$ coincides with $\mathcal{G}(\oplus u_j)$. Therefore it is a harmonic extension of Green's integral $\mathcal{G}(\oplus u_j)$ from O^+ to the whole domain O.

Now using Theorem 2.8.1.2 we conclude that Problem 2.8.1.1 is solvable. This proves the theorem. $\square$

Let us give now the corresponding variant of Carleman's formula. For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x, y) = \varphi_n(x - y) - \varphi_n(y) + \sum_{\nu=1}^{N} \sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{n + 2\nu - 2} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.$$

LEMMA 2.8.2.5. *For any number $N = 1, 2, ...,$ the kernel $\mathfrak{C}^{(N)}$ is harmonic with respect to x and y for all $y \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. Follows from the properties of the $\varphi_n(x-y)$ and the polynomials $h_\nu^{(i)}(y)$. $\square$

We note that since $\mathfrak{C}^{(N)}$ is a "remainder" summand in the formula (2.8.2.1), $\mathfrak{C}^{N)}(x, y) \to 0$ $(N \to \infty)$, together with all its derivatives uniformly on compact subsets of the cone $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

THEOREM 2.8.2.6 (CARLEMAN'S FORMULA). *For any harmonic function $u \in C^1_{loc}(D \cup S)$ whose restriction to S is summable there, the following formula holds*
(2.8.2.10)

$$u(x) = \lim_{N\to\infty} \int_S \left(u(y) \frac{\partial \mathfrak{C}^{(N)}(x, y)}{\partial n_y} - \frac{\partial u(y)}{\partial n_y} \mathfrak{C}^{(N)}(x - y) \right) ds(y) \ (x \in D).$$

PROOF. This is similar to the proof of Theorem 2.6.6. $\square$

Remark 2.8.2.7. As in Theorem 2.6.6, the convergence of the limit in (2.8.2.10) is uniform on compact subsets of the domain D together with all its derivatives.

EXAMPLE 2.8.2.8. If $n = 2$ then O is the circle in $\mathbb{R}^2$. As a system of spherical harmonics we can take the system $h_0^{(1)} = 1/\sqrt{2\pi}$, $h_\nu^{(1)} = (x_1 + \sqrt{-1}x_2)^\nu/\sqrt{2\pi}$, $h_\nu^{(2)} = (x_1 - \sqrt{-1}x_2)^\nu/\sqrt{2\pi}$ with $x = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\mathfrak{C}^{(N)}(x, y) = \frac{1}{2\pi} ln|x - y| - \frac{1}{2\pi} ln|y| + \frac{1}{2\pi} Re \left(\sum_{\nu=1}^{N} \left(\frac{x_1 + \sqrt{-1}x_2}{y_1 + \sqrt{-1}y_2} \right)^\nu \frac{1}{\nu} \right)$$

where $y = (y_1, y_2) \in \mathbb{R}^2$ and $Re(c)$ stands for the real part of the complex number c.

2.8.3 Example for the Cauchy problem for the Laplace operator in a shell in $\mathbb{R}^n$.

In this section we consider the Cauchy problem for harmonic functions in a shell D in $\mathbb{R}^n$ whose exterior surface is a smooth closed hypersurface S in $\mathbb{R}^n$ and interior surface is a sphere $\partial B_r = \{x \in \mathbb{R}^n : \ |x| = r\}$ with centre at zero and radius $0 < r < \infty$, with the Cauchy data on S (cf. [Sh5]).

For this purpose we will take as the domain O a shell $G(r, R) = \{x \in \mathbb{R}^n : \ r < |x| < R\}$ $(0 < r < R < \infty)$, with sufficiently big R, and, as in 2.8.2, we will use the spherical harmonics $h_\nu^{(i)}$.

LEMMA 2.8.3.1. *For any shell* $G(r_1, r_2)$ $(0 < r_1 < r_2 < \infty)$ *we have*

$$\int_{G(r_1,r_2)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} \frac{h_\mu^{(j)}(x)}{|x|^{n+2\mu-2}}\, dx = \begin{cases} \frac{r_2^{4-n-2\nu} - r_1^{4-n-2\nu}}{4-n-2\nu}, & \nu = \mu, \text{ and } i = j \\ 0, \ \nu \neq \mu \text{ or } j \neq i, \end{cases}$$

with $\nu = 1, 2, \ldots$*, except the case* $n = 2$*,* $\mu = \nu = 2$*;*

$$\int_{G(r_1,r_2)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} \frac{h_\mu^{(j)}(x)}{|x|^{n+2\mu-2}}\, dx = \begin{cases} \frac{r_2^{4-n-2\nu} - r_1^{4-n-2\nu}}{4-n-2\nu}, & \nu = \mu, \text{ and } i = j \\ 0, \ \nu \neq \mu \text{ or } j \neq i, \end{cases}$$

with $\nu = 1, 2, \ldots$*, except the case* $n = 2$*,* $i = j$*,* $\mu = \nu = 2$*;*

$$\int_{G(r_1,r_2)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}\, \varphi_n(x)\, dx = 0 \ (\nu = 1, 2, \ldots);$$

$$\int_{G(r_1,r_2)} h_\nu^{(i)}(x)\, h_\mu^{(j)}(x)\, dx = \begin{cases} (r_2^{n+2\nu} - r_1^{n+2\nu})/(n+2\nu), \ \nu = \mu, \text{ and } i = j, \\ 0, \ \nu \neq \mu \text{ or } j \neq i, \end{cases}$$

with $\nu = 0, 1, \ldots.$

PROOF. Let us prove the first equality (the proofs of the others are similar).

$$\int_{G(r_1,r_2)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} \frac{h_\mu^{(j)}(x)}{|x|^{n+2\mu-2}}\, dx = \int_{r_1}^{r_2} dt \int_{|x|=t} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} \frac{h_\mu^{(j)}(x)}{|x|^{n+2\mu-2}} d\sigma(x) =$$

$$= \int_{r_1}^{r_2} t^{3-\nu-\mu-n} dr \int_{|x|=1} \overline{h_\nu^{(i)}(x)} h_\mu^{(j)}(x) d\sigma(x) =$$

$$= \begin{cases} r_2^{4-n-2\nu} - r_1^{4-n-2\nu}/(4-n-2\nu) \ \nu = \mu, \text{ and } i = j, \\ 0, \ \nu \neq \mu \text{ or } j \neq i. \end{cases} \qquad \square$$

Now, using Lemmata 2.8.2.1, 2.8.3.1, we can write the Laurent series for harmonic functions in a shell $G(r_1, r_2) = \{x \in \mathbb{R}^n : \ r_1 < |x| < r_2\}$ (cf. [T4], Corollary 8.11).

PROPOSITION 2.8.3.2. *Every function* $u \in C^1(\overline{G(r_1, r_2)})$*, harmonic in* $G(r_1, r_2)$*, can be expanded as follows:*

$$u(x) = \sum_{\nu=0}^{\infty} \sum_{i=1}^{J(\nu)} a_\nu^{(i)} h_\nu^{(i)}(x) + b_0 \varphi_n(x) + \sum_{\nu=1}^{\infty} \sum_{i=1}^{J(\nu)} b_\nu^{(i)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}, \tag{2.8.3.1}$$

where the series converge absolutely together with all the derivatives uniformly on compact subsets of $G(r_1, r_2)$ *and the coefficients* $a_\nu^{(i)}$*,* $b_\nu^{(i)}$ *are uniquely defined.*

PROOF. Let $u \in C^1(\overline{G(r_1, r_2)})$ be harmonic in $G(r_1, r_2)$. It is known that in this case u can be can be represented in $G(r_1, r_2)$ by Green's formula. Replacing the fundamental solution $\varphi_n(x - y)$ in this Green's formula by decomposition (1.2), we obtain that

$$u(x) = \int_{|y|=r_2} \left(u(y) \frac{\partial \varphi_n(x-y)}{\partial n_y} - \frac{\partial u(y)}{\partial n} \varphi_n(x-y) \right) ds(y) +$$

$$+\int_{|y|=r_1}\left(u(y)\frac{\partial\varphi_n(x-y)}{\partial n_y}-\frac{\partial u(y)}{\partial n}\varphi_n(x-y)\right)ds(y)=$$

$$=\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)}a_\nu^{(i)}h_\nu^{(i)}(x)+b_0\varphi_n(x)+\sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)}b_\nu^{(i)}\frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}},\ (r_1<|x|<r_2) \tag{2.8.3.2}$$

where the coefficients $a_\nu^{(i)}$, $b_\nu^{(i)}$ are defined by the following formulae:

$$a_\nu^{(i)}=\begin{cases}\frac{-1}{n+2\nu-2}\int_{|y|=r_2}\left(u(y)\frac{\partial}{\partial n}\left(\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}\right)-\frac{\partial u(y)}{\partial n}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}\right)ds(y)\ (\nu=1,2,...),\\ \sqrt{\sigma_n}\int_S\left(u(y)\frac{\partial\varphi_n(y)}{\partial n}-\frac{\partial u(y)}{\partial n}\varphi_n(y)\right)ds(y),\ \nu=0,\end{cases}$$

$$b_\nu^{(i)}=\begin{cases}\frac{-1}{n+2\nu-2}\int_{|y|=r_1}\left(u(y)\frac{\partial h_\nu^{(i)}(y)}{\partial n}-\frac{\partial u(y)}{\partial n}h_\nu^{(i)}(y)\right)ds(y)\ (\nu=1,2,...),\\ -\int_{|y|=r_1}\frac{\partial u(y)}{\partial n}ds(y)\ \nu=0,\end{cases}$$

In order to prove that coefficients $a_\nu^{(i)}$, $b_\nu^{(i)}$ are uniquely defined we note that, according to Lemma 7.20 of [T4], every harmonic in the shell $G(r_1,r_2)$ function u can be represented in the form

$$u(x)=u^+(x)+u^-(x)$$

where u^+, u^- are uniquely defined such that u^+ is harmonic in the ball B_{r_2} and u^- is harmonic in $\mathbb{R}^n\backslash\overline{B}_{r_1}$ and regular at infinity. Clearly, in our case

$$u^+=\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)}a_\nu^{(i)}h_\nu^{(i)}(x),$$

$$u^-=b_0\varphi_n(x)+\sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)}b_\nu^{(i)}\frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}.$$

Now, using Lemma 2.8.3.1, we see that the coefficients $a_\nu^{(i)}$, $b_\nu^{(i)}$ are uniquely defined. □

Our principal results in this section will be formulated in terms of the coefficients

$$k_\nu^{(i)}=\begin{cases}\frac{-1}{n+2\nu-2}\int_S\left(u_0(y)\frac{\partial h_\nu^{(i)}(y)}{\partial n}-u_1(y)h_\nu^{(i)}(y)\right)ds(y)\ (\nu=1,2,...),\\ -\int_{|y|=r_1}u_1(y)ds(y)\ (\nu=0).\end{cases}$$

THEOREM 2.8.3.3. *Let u_0, $u_1\in L^1(S)$. Then, for Problem 2.8.1.1 to be solvable, it is necessary and sufficient that*

$$\limsup_{\nu\to\infty}\max_i\sqrt[\nu]{|k_\nu^{(i)}|}\le r \tag{2.8.3.3}$$

PROOF. It is similar to the proof of Theorem 2.8.2.4. □

Let us to obtain Carleman's formula for solutions of Problem 2.8.1.1 in this case.

For each number $N=1,2...$ we consider the kernel $\mathfrak{C}^{(N)}(x,y)$ defined, for all $y\neq 0$ off the diagonal $\{x=y\}$, by the equality

$$\mathfrak{C}^{(N)}(x,y)=\varphi_n(x-y)-\varphi(x)+\sum_{\nu=1}^{N}\sum_{i=1}^{J(\nu)}\frac{h_\nu^{(i)}(y)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}.$$

PROPOSITION 2.8.3.4. *For any number $N = 1, 2, ...$, the kernel $\mathfrak{C}^{(N)}$ is harmonic with respect to x and y for all $x \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. Follows from the properties of the $\varphi n(x-y)$ and the polynomials $h_\nu^{(i)}(y)$.
$\square$

THEOREM 2.8.3.5. (CARLEMAN'S TYPE FORMULA). *For any harmonic function $u \in C(D \cup S)$ whose restriction to S is summable there, the following formula holds*

$$u(x) = \lim_{N\to\infty} \int_S \left(u(y) \frac{\partial \mathfrak{C}^{(N)}(x,y)}{\partial n_y} - \frac{\partial u(y)}{\partial n} \mathfrak{C}^{(N)}(x,y) \right) ds(y) \ (x \in D). \tag{2.8.3.4}$$

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2.9. Example for the Lamé type system in $\mathbb{R}^n$

2.9.1 Solvability condition for the Cauchy problem for the Lamé type system in $\mathbb{R}^n$ in terms of Green's integral.

In this section we study the Cauchy problem for the system

$$\mathfrak{L} = \mu\, \Delta_n + (\lambda + \mu)\, \nabla_n\, div_n,$$

with constants $\mu \neq 0$, $\lambda \neq -2\mu$.

In Elasticity Theory ($n = 2, 3$), with *Lamé constants* λ, μ, this system is known as the *Lamé system*.

More exactly, denoting by $\nu_j(x)$ the j-th component of the unit outward normal vector $\nu(x)$ to ∂D at the point x, by $\frac{\partial}{\partial\nu}$ the normal derivative with respect to ∂D and by T the *stress operator*, i.e the matrix $T(x, D) = (T_{ij}(x, D)_{i,j=1,2,...,n}$ with components

$$T_{ij}(x, D) = \mu\, \delta_{ij}\, \frac{\partial}{\partial\nu} + \lambda\, \nu_i(x)\, \frac{\partial}{\partial x_j} + \mu\, \nu_j(x)\, \frac{\partial}{\partial x_i} \ \ (i, j = 1, ..., n),$$

we consider the following problem.

PROBLEM 2.9.1.1. *Let vector-functions $u_0(x) = (u_0^1(x), ..., u_0^n(x))^T \in [C^1(S)]^n$ and $u_1(x) = (u_1^1(x), ..., u_1^n(x))^T \in [C(S)]^n$, be given. It requires to find (if possible) a vector-function $u(x) \in [C^1(D \cup S) \cap C^2(D)]^n$ such that*

$$\begin{cases} \mathfrak{L}u = f \ in\ D, \\ u_{|S} = u_0, \\ (Tu)_{|S} = u_1. \end{cases}$$

Since $\mu \neq 0$, $\lambda \neq -2\mu$, and

$$det\, \sigma(\mathfrak{L})(x, \zeta) = \mu^{n-1}(\lambda + 2\mu)|\zeta|^{2n},$$

the Lamé type system $\mathfrak{L}$ is elliptic. One easily sees that the boundary system $\{B_0 = Id_n, B_1 = T\}$ is a Dirichlet system of the first order on ∂D ($det\, \sigma(T)(x, d\rho) = \mu^{n-1}(\lambda + 2\mu)|d\rho|^{2n}$).

By direct calculation one obtains

Lemma 9.1.1.2. *The matrix* $\Phi(x) = (\Phi_{ij}(x))_{i,j=1,2,\dots,n}$ *with components*

$$\Phi_{ij}(x) = \frac{1}{2\mu(\lambda+2\mu)} \left(\delta_{ij}\,(\lambda+3\mu)\varphi_n(x) - (\lambda+\mu)\,x_j\,\frac{\partial}{\partial x_i}\varphi_n(x) \right) \quad (i,j = 1,2,\dots,n),$$

where δ_{ij} *is the Kronecker delta, is a fundamental solution of convolution type for the homogeneous Lamé type system* $\mathfrak{L}$.

The matrix Φ is called the *Kelvin-Somigliana matrix* for $n = 3$ (see, for example, [Kup]).

Green's formula (1.3.1) in this case is the Somigliana formula (see, for example, [Kup]):

(9.1.1.1)

$$\int_{\partial D} \big(T(y,D)\Phi(x-y)^T u(y) - \Phi(x-y)T(y,D)u(y)\big)\,ds(y) = \begin{cases} u(x), & x \in D, \\ 0, & x \notin \overline{D}. \end{cases}$$

Assume that the functions u_0, u_1 are summable on S. Then the corresponding Green's integral is well defined:

(2.9.1.2)

$$\mathcal{G}(\oplus u_j)(x) = \int_S \big((T(y,D)\Phi(x-y))^T\,u_0(y) - \Phi(x-y)\,u_1(y)\big)\,ds(y) \ (x \in O\backslash S).$$

It is clear that $\mathcal{G}(\oplus u_j)$ is a solution of the homogeneous Lamé type system everywhere outside of S; let $\mathcal{G}(\oplus u_j)^{\pm} = \mathcal{G}(\oplus u_j)_{|O^{\pm}}$.

Theorem 2.4.2 and Lemma 1.3.4 imply the following result.

Theorem 2.9.1.3. *Let* $S \in C^2$, $u_0 \in [C^1(S)]^n$ *and* $f^1 \in [C^0(S)]^n$ *be summable vector-functions on* S. *Then, for Problem 2.9.1.1 to be solvable, it is necessary and sufficient that the integral* $\mathcal{G}(\oplus u_j)^+$ *harmonically extends from* O^+ *to the domain* O.

Proof. See also paper [Sh4].

In the next two subsections we will use decomposition (2.8.2.1) to obtain Carleman's type formula in domains of special types for solutionsof the system $\mathfrak{L}$.

2.9.2 Example for the Cauchy problem for the Lamé system in a part of a ball in $\mathbb{R}^n$.

Let $O = B_R$ be the ball with centre at zero and radius $0 < R < \infty$, and S be a closed smooth hypersurface dividing it into 2 connected components (O^+ and $O^- = D$) in such away that $0 \in O^+$, and oriented as the boundary of O^-. Using Lemma 2.8.2.1, we obtain the following decompositions for the fundamental solution Φ of the homogeneous Lamé system.

Lemma 2.9.2.1. *The fundamental solution* $\Phi(x-y)$ *of the Lamé type system* $\mathfrak{L}$ *can be expanded as follows:*

$$\Phi(x-y) = \sum_{\nu=0}^{\infty} \Phi^{(\nu)}(x,y)$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |y| > |x|\}$ *and* $\Phi^{(\nu)}(x,y)$

$(\nu \geq 0)$ are matrices with components $\Phi^{(\nu)}_{kl}(x,y)$ $(k,l = 1,2,...,n)$:

$$\Phi^{(0)}_{kl}(x,y) = \varphi_n(y)\frac{(\lambda+3\mu)\delta_{kl}}{2\mu(\lambda+2\mu)} - \sum_{i=1}^{n} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{y_l\,\overline{h^{(i)}_1(y)}}{n\,|y|^n}\frac{\partial h^{(i)}_1(x)}{\partial x_k},$$

$$\Phi^{(\nu)}_{kl}(x,y) = -\sum_{i=1}^{J(\nu)} \frac{\overline{h^{(i)}_\nu(y)}}{|y|^{n+2\nu-2}}\left(\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\frac{\delta_{kl}\,h^{(i)}_\nu(x)}{(n+2\nu-2)} - \right.$$

$$\left. - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial h^{(i)}_\nu(x)}{\partial x_k}\frac{x_l}{(n+2\nu-2)}\right) -$$

$$- \sum_{i=1}^{J(\nu+1)} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{\overline{h^{(i)}_{\nu+1}(y)}}{|y|^{n+2\nu}}\frac{y_l}{(n+2\nu)}\frac{\partial h^{(i)}_{\nu+1}(x)}{\partial x_k}\ (\nu \geq 1). \tag{2.9.2.1}$$

LEMMA 2.9.2.2. *For $\nu = 1,2,...$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n\backslash\{0\}$, we have*

$$\mathfrak{L}(x)\Phi^{(\nu)}_{kl}(x,y) = 0, \Delta^2_n(y)\Phi^{(\nu)}_{kl}(x,y) = 0.$$

PROOF. Due to the harmonicity of the polynomials $h^{(i)}_\nu$, the matrix, whose components are formed by the first sum in the right hand side of (2.9.2.1), is a solution of the homogeneous Lamé type system.

On the other hand, the matrix, whose components are formed by the second sum in the right hand side of (2.9.2.1), is equal to

$$\sum_{i=1}^{J(\nu+1)} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{\overline{h^{(i)}_{\nu+1}(y)}}{|y|^{n+2\nu}}\frac{\nabla_x h^{(i)}_{\nu+1}(x) y^T}{(n+2\nu)}.$$

Therefore, because of the harmonicity of the polynomials $h^{(i)}_\nu$, it is a solution of the homogeneous Lamé type system too.

The biharmonicity of $\Phi^{(\nu)}$ is obvious. $\square$

We obtain now a decomposition of the vector-function $\mathcal{G}(\oplus u_j)$ in a neighbourhood of origin.

LEMMA 2.9.2.3. *Let $0 < \rho < dist(0,S)$ be fixed, so that the ball $B_\rho \Subset O^+$. Then*

$$\mathcal{G}(\oplus u_j)^+(x) = \sum_{\nu=0}^{\infty} H_\nu(x)\ (x \in B_\rho), \tag{2.9.2.2}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_ρ and H_ν are homogeneous polynomials of degree ν satisfying $\mathfrak{L}\,H_\nu = 0$ in $\mathbb{R}^n$:

$$H_\nu(x) = \int_S \Big((T(y,D)\Phi^{(\nu)}(x,y))^T\, u_0(y) - \Phi^{(\nu)}(x,y)\, u_1(y)\Big)\, ds(y).$$

PROOF. Since $0 \notin S$,

$$\max_{x \in B_\rho, y \in \overline{S}} \frac{|x|}{|y|} \le q < 1.$$

Then, using estimate (2.8.2.6), one easily obtains that

$$|H_\nu(x)| \le C\, q^{2\nu} (\nu+1)^{n-2} \ (x \in B_\rho,\ \nu \ge 0) \tag{2.9.2.3}$$

with a constant $C > 0$ which depends on u_0, u_1 and does not depend on ν and x. Estimate (2.9.2.3) implies that the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_ρ. Now, using formula (2.9.1.2) and Lemmata 2.9.2.1 and 2.9.2.2, we conclude that the statement of the lemma holds. □

PROPOSITION 2.9.2.4. *Let $S \in C^2$, $u_0 \in [C^1(S)]^n$ and $u_1 \in [C(S)]^n$ be summable vector-functions on S. Then, for Problem 2.9.1.1 to be solvable, it is necessary and sufficient that the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_R.*

PROOF. Necessity. Let Problem 2.9.1.1 be solvable. Then Theorem 2.9.1.3 imply that the integral $\mathcal{G}(\oplus u_j)^+$ on the domain O^+ extends to a solution $\mathcal{F}$ of the homogeneous Lamé system in B_R.

We fix $0 < r_0 < R$. It is clear that $\mathcal{F} \in [C^1(\overline{B}_{r_0})]^n$. Hence, it represents in the ball B_{r_0} in the following way (see formula (2.9.1.1)):

$$\mathcal{F}(x) = \int_{\partial B_{r_0}} \big((T(y,D)\Phi(x-y))^T \mathcal{F}(y) - \Phi(x-y)\, T(y,D)\mathcal{F}(y) \big)\, ds(y).$$

Substituting instead of Φ its decomposition obtained in Lemma 2.9.2.1 and arguing in the same way as in Lemma 2.9.2.3, we obtain that, for $x \in B_r$ $(0 < r < r_0)$,

$$\mathcal{F}(x) = \sum_{\nu=0}^{\infty} \widetilde{H}_\nu(x) \ (x \in B_r), \tag{2.9.2.4}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_r and $\widetilde{H}_\nu$ are homogeneous polynomials of degree ν and solutions of the homogeneous Lamé type system $\mathfrak{L}$ in $\mathbb{R}^n$:

$$\widetilde{H}_\nu(x) = \int_{\partial B_{r_0}} \Big((T(y,D)\Phi^{(\nu)}(x,y))^T \mathcal{F}(y) - \Phi^{(\nu)}(x,y)\, T(y,D)\mathcal{F}(y) \Big)\, ds(y).$$

Comparing (2.9.2.2) and (2.9.2.4) we find that

$$D^\alpha H_\nu = D^\alpha \widetilde{H}_\nu = D^\alpha(\mathcal{G}(\oplus u_j)^+)(0) \ \ (|\alpha| = \nu,\ \nu = 0, 1, \dots).$$

Because H_ν, $\widetilde{H}_\nu$ are homogeneous , we conclude that, for $x \in \mathbb{R}^n$,

$$H_\nu(x) = \widetilde{H}_\nu(x) \ \ (\nu = 0, 1, \dots).$$

Therefore the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_r.

Since $0 < r_0 < R$ is arbitrary then the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_R, which was to be proved.

Sufficiency. Let the series $\mathcal{F}(x) = \sum_{\nu=0}^{\infty} H_\nu(x)$ converge absolutely together with all the derivatives uniformly on compact subsets of the ball B_R. Since the polynomials H_ν are solutions of the homogeneous Lamé type system in $\mathbb{R}^n$, by Stiltjes-Vitali theorem we conclude that $\mathcal{F}$ satisfies $\mathfrak{L}\mathcal{F} = 0$ in B_R.

It is easy to see from Lemma 2.9.2.3, that in the ball B_r the vector-function $\mathfrak{F}$ coincides with $\mathcal{G}(\oplus u_j)^+$. Now using Theorem 2.9.1.3 we see that Problem 2.9.1.1 is solvable. This proves the proposition. □

Proposition 2.9.2.4 can be used to prove Carleman's formula for determination of a solution u of the Lamé type system in B_R by its Cauchy data on S.

For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x, y) = \Phi(x - y) - \sum_{\nu=0}^{N} \Phi^{(\nu)}(x, y).$$

PROPOSITION 2.9.2.5. *For any number $N = 1, 2, ...$, the kernel $\mathfrak{C}^{(N)}$ satisfies the equations $\mathfrak{L}(x)\mathfrak{C}^{(N)}(x, y) = 0$, $\Delta_n^2(y)\mathfrak{C}^{(N)}(x, y) = 0$ for all $y \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. Follows from the properties of the $\Phi(x - y)$ and Lemma 2.9.2.1. □

THEOREM 2.9.2.6 (CARLEMAN'S FORMULA). *Let $S \in C^2$. Then, for any solution $u \in [C^1_{loc}(D \cup S)]^n$ of the Lamé type system $\mathfrak{L}$ such that $u_{|S}$ and $(Tu)_{|S}$ are summable on S, the following formula holds:*

$$u(x) = \lim_{N\to\infty} \int_S \Big((T(y, D)\mathfrak{C}^{(N)}(x, y))^T u_0(y) - \mathfrak{C}^{(N)}(x, y)\, u_1(y) \Big)\, ds(y). \tag{2.9.2.5}$$

PROOF. This is similar to the prooh of Theorem 2.6.6 (see also [Sh4]). □.

A Carleman formula for solutions of the Lamé system in $\mathbb{R}^3$ was established in [Ma] for specific choices of D, for example if it is bounded by part of the surface of a cone $\mathcal{K}$ and a smooth piece of S in the interior of $\mathcal{K}$, or if it is a relatively compact domain in $\mathbb{R}^3$ whose boundary consists of a piece of the plane $\{x_3 = 0\}$ and a smooth surface S lying in the half-space $\{x_3 > 0\}$.

Remark 2.9.2.7. As in Theorem 2.6.6, the convergence of the limit in (2.9.2.5) is uniform on compact subsets of the domain D together with all its derivatives.

2.9.3 Example for the Cauchy problem for the Lamé system in a shell in $\mathbb{R}^n$.

Let us consider now the situation where D is a shell $\mathbb{R}^n$ whose exterior surface is a smooth closed hypersurface S in $\mathbb{R}^n$ and interior surface is a sphere $\partial B_r = \{x \in \mathbb{R}^n : |x| = r\}$ with centre at zero and radius $0 < r < \infty$, with the Cauchy data on S.

As in 2.8.3, we take as the domain O a shell $G(r, R) = \{x \in \mathbb{R}^n : r < |x| < R\}$ $(0 < r < R < \infty)$, with sufficiently big R.

LEMMA 2.9.3.1. *The fundamental solution $\Phi(x-y)$ of the Lamé type system $\mathfrak{L}$ can be expanded as follows:*

$$\Phi(x-y) = \Phi(x) + \sum_{\nu=1}^{\infty} \widetilde{\Phi}^{(\nu)}(x,y)$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| > |y|\}$ and $\Phi^{(\nu)}(x,y)$ $(\nu \geq 0)$ are matrices with components $\Phi^{(\nu)}_{kl}(x,y)$ $(k,l = 1,2,...,n)$:

$$\widetilde{\Phi}^{(1)}_{kl}(x,y) = -\sum_{i=1}^{(n)} \frac{h_1^{(i)}(y)}{(n+2)} \left(\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \frac{\delta_{kl}\, \overline{h_1^{(i)}(x)}}{|x|^{n+2}} - \right.$$

$$\left. - \frac{(\lambda+\mu)x_l}{2\mu(\lambda+2\mu)} \frac{\partial}{\partial x_k} \left(\frac{\overline{h_1^{(i)}(x)}}{|x|^{n+2}} \right) \right) - \frac{(\lambda+\mu)y_l}{2\mu(\lambda+2\mu)} \frac{\partial g(x)}{\partial x_k},$$

$$\widetilde{\Phi}^{(\nu)}_{kl}(x,y) = -\sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(y)}{(n+2\nu-2)} \left(\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \frac{\delta_{kl}\, \overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} - \right.$$

$$\left. - \frac{(\lambda+\mu)x_l}{2\mu(\lambda+2\mu)} \frac{\partial}{\partial x_k} \left(\frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}} \right) \right) -$$

$$(2.9.3.1) \qquad - \sum_{i=1}^{J(\nu-1)} \frac{(\lambda+\mu)y_l}{2\mu(\lambda+2\mu)} \frac{h_{\nu-1}^{(i)}(y)}{(n+2\nu-4)} \frac{\partial}{\partial x_k} \left(\frac{\overline{h_{\nu-1}^{(i)}(x)}}{|x|^{n+2\nu-4}} \right) \quad (\nu \geq 2).$$

LEMMA 2.9.3.2. *For $\nu = 1,2,...$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n \backslash \{0\}$ we have*

$$\mathfrak{L}_x \widetilde{\Phi}^{(\nu)}_{kl}(x,y) = 0, \ \Delta_y^2 \widetilde{\Phi}^{(\nu)}_{kl}(x,y) = 0.$$

PROOF. It is similar to the proof of Lemma 2.9.2.2. □

Now we obtain an analogue of the Laurent series for solutions of the Lamé system $\mathfrak{L}$ functions (cf. [T4], §7).

PROPOSITION 2.9.3.3. *Every function $u \in C^1(\overline{G(r_1,r_2)})$, satisfying $\mathfrak{L}u = 0$ in $G(r_1,r_2)$, can be expanded as follows:*

$$(2.9.3.2) \qquad u(x) = \sum_{\nu=0}^{\infty} H_\nu^+(x) + A_0\Phi(x) + \sum_{\nu=1}^{\infty} H_\nu^-(x)$$

where

(1) *the series converge absolutely together with all the derivatives uniformly on compact subsets of $G(r_1,r_2)$;*
(2) *H_ν^+ are n-vectors of homogeneous polynomials of degree ν with $\mathfrak{L}H_\nu^+ = 0$ in $\mathbb{R}^n$;*
(3) *H_ν^- are n-vectors of homogeneous functions of degree $2-n-\nu$ with $\mathfrak{L}H_\nu^+ = 0$ in $\mathbb{R}^n\backslash\{0\}$;*
(4) *A_0 and $H_\nu^\pm$ are uniquely defined.*

PROOF. Let $u \in C^1(\overline{G(r_1, r_2)})$ be a solution of the Lamé type system $\mathfrak{L}$ in $G(r_1, r_2)$. Using Lemma (2.9.1.1) we represent u in $G(r_1, r_2)$ by Somigliana formula:

$$u(x) = \int_{|y|=r_1} \left((T(y,D)\Phi(x-y))^T u(y) - \Phi(x-y)\, T(y,D)u(y) \right) ds(y) +$$

$$+ \int_{|y|=r_2} \left((T(y,D)\Phi(x-y))^T u(y) - \Phi(x-y)\, T(y,D)u(y) \right) ds(y) \ (r_1 < |x| < r_2).$$

Replacing the fundamental solution $\Phi(x-y)$ in this formula by decompositions, obtained in Lemmata 2.9.2.1, 2.9.3.1, we see that

$$u(x) = \sum_{\nu=0}^{\infty} H_\nu^+(x) + A_0\Phi(x) + + \sum_{\nu=1}^{\infty} H_\nu^-(x) \ (r_1 < |x| < r_2),$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the shell $G(r_1, r_2)$ and A_0, $H_\nu^\pm$ are defined by the following formulae:

$$A_0(x) = \int_S T(y,D)u(y) ds(y),$$

$$H_\nu^-(x) = \int_{|y|=r_1} \Big((T(y,D)\Phi^{(\nu)}(x,y))^T u(y) - \Phi^{(\nu)}(x,y)\, T(y,D)u(y) \Big) ds(y) \ (\nu \geq 1),$$

$$H_\nu^+(x) = \int_{|y|=r_2} \Big((T(y,D)\widetilde{\Phi}^{(\nu)}(x,y))^T u(y) - \widetilde{\Phi}^{(\nu)}(x,y)\, T(y,D)u(y) \Big) ds(y).$$

Clearly (3) and (4) follows from Lemmata 2.9.2.2, 2.9.3.2 and properties of the polynomials $h_\nu^{(i)}$.

Let us prove now that A_0 and $H_\nu^\pm$ are uniquely defined. As in the proof of Proposition 1.4, using Lemma 7.20 of [T4], we see that any solution u of the Lamé system $\mathfrak{L}$ can be represented in the form

$$u(x) = u^+(x) + u^-(x)$$

where $u^\pm$ are uniquely defined such that u^+ is a solution of the Lamé type system $\mathfrak{L}$ in the ball B_{r_2} and u^- is a solution of the Lamé type system $\mathfrak{L}$ in $\mathbb{R}^n \backslash \overline{B}_{r_1}$.

Obviously

$$u^+ = \sum_{\nu=0}^{\infty} H_\nu^+(x),$$

$$u^- = A_0\Phi(x) + \sum_{\nu=1}^{\infty} H_\nu^-(x).$$

Let us assume that there exists another decomposition

$$u^+(x) = \sum_{\nu=0}^{\infty} \widetilde{H}_\nu^+(x),$$

$$u^-(x) = \widetilde{A_0}\Phi(x) + \sum_{\nu=1}^{\infty} \widetilde{H}_\nu^-(x) \ (r_1 < |x| < r_2).$$

Then

$$D^\alpha H_\nu^+ = D^\alpha \widetilde{H}_\nu^+ = D^\alpha u^+(0) \ \ (|\alpha| = \nu, \ \nu = 0, 1, ...).$$

Because H_ν, $\widetilde{H}_\nu$ are homogeneous polynomials of degree $\nu \geq 0$, we conclude that, for $x \in \mathbb{R}^n$,

$$H_\nu(x) = \widetilde{H}_\nu(x) \ \ (\nu = 0, 1, ...).$$

On the other hand, for $x \in \partial B_1$ ($|x| = 1$) we have

$$A_0 - \widetilde{A_0} = \lim_{\lambda \to \infty} \frac{\sum_{\nu=1}^{\infty} (H_\nu^-(\lambda x) - \widetilde{H}_\nu^-(\lambda x)}{\Phi(\lambda x)} = 0.$$

Arguing in a similar way one obtains that

$$H_\nu^-(x) = \widetilde{H}_\nu^-(x) \ (\nu \geq 1). \quad \square$$

LEMMA 2.9.3.4. *Let $r < \rho_1 < \rho_2 < R$ be fixed, so that the shell $G(\rho_1, \rho_2) \Subset G^+$. Then*

$$\mathcal{G}(\oplus u_j)^+(x) = \sum_{\nu=0}^{\infty} H_\nu(x) \ (x \in G(\rho_1, \rho_2)), \tag{2.9.3.3}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the shell $G(\rho_1, \rho_2)$ and H_ν are n-vectors of homogeneous functions of degree $2 - n - \nu$, satisfying $\mathfrak{L} H_\nu = 0$ in $\mathbb{R}^n \backslash \{0\}$:

$$H_0(x) = \Phi(x) \int_S u_1(y) ds(y),$$

$$H_\nu(x) = \int_S \Big((T(y, D)\Phi^{(\nu)}(x, y))^T u_0(y) - \Phi^{(\nu)}(x, y) \, u_1(y) \Big) \, ds(y) \ (\nu \geq 1).$$

PROOF. Since $G(\rho_1, \rho_2) \Subset G^+$,

$$\max_{x \in G(\rho_1, \rho_2), y \in \overline{S}} \frac{|y|}{|x|} \leq q < 1.$$

Hence it follows from (2.9.1.1), and the proof is similar to the proof of Lemma 2.9.3.3. $\square$

PROPOSITION 2.9.3.5. *Let $S \in C^2$, $u_0 \in [C^1(S)]^n$ and $u_1 \in [C(S)]^n$ be summable vector-functions on S. Then, for Problem 2.9.1.1 to be solvable, it is necessary and sufficient that the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the shell $G(r, R)$.*

PROOF. Follows from Lemmata 2.9.3.3 and 2.9.3.4, as Proposition 2.9.2.4 from Lemma 2.9.2.3. $\square$

Proposition 2.9.3.5 can be used to prove Carleman's formula for determination of a solution u of the Lamé type system $\mathfrak{L}$ in D by its Cauchy data on S.

For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x, y) = \Phi(x - y) - \Phi(x) - \sum_{\nu=1}^{N} \Phi^{(\nu)}(x, y).$$

PROPOSITION 2.9.3.6. *For any number* $N = 1, 2, ...,$ *the kernel* $\mathfrak{C}^{(N)}$ *satisfies the equations* $\mathfrak{L}_x \mathfrak{C}^{(N)}(x, y) = 0$, $\Delta_y^2 \mathfrak{C}^{(N)}(x, y) = 0$ *for all* $x \neq 0$ *off the diagonal* $\{x = y\}$.

PROOF. Follows from the properties of the $\Phi(x - y)$ and Lemma 2.9.3.2. $\square$

The following formula produces rather explicitly a way to obtain a solution of the Lamé type system $\mathfrak{L}$ by successive approximations (see [Sh5]).

THEOREM 2.9.3.7 (CARLEMAN'S TYPE FORMULA). *Let* $S \in C^2$. *Then, for any solution* $u \in [C^1(D \cup S)]^n$ *of the Lamé type system* $\mathfrak{L}$ *such that* $u_{|S}$ *and* $(Tu)_{|S}$ *are summable on* S, *the following formula holds:*

$$u(x) = \lim_{N \to \infty} \int_S \Big((T(y, D)\mathfrak{C}^{(N)}(x, y))^T u(y) - \mathfrak{C}^{(N)}(x, y)\, Tu(y) \Big)\, ds(y). \tag{2.9.3.2}$$

Remark 2.9.3.8. The limit in (2.9.3.2) is uniform on compact subsets of $D \cup S$ together with all its derivatives.

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§2.10. Reduction of the Cauchy problem for systems with injective symbols to the Cauchy problem for determined elliptic systems

We continue to consider the Cauchy problem for solutions of the system $Pu = 0$ where P is an elliptic operator on an open set X in $\mathbb{R}^n$.

PROBLEM 2.10.1. *Let* u_j $(0 \leq j \leq p - 1)$ *be sections of the bundles* F_j *over an open set* S. *It is required to find a solution* $u \in S_P^f(D)$ *of finite order of growth such that the expressions* $B_j u$ $(0 \leq j \leq p - 1)$ *have weak limit values on* S *coinciding with* u_j.

In this and in the following 3 sections of this chapter we assume that the coefficients of the operator P are real analytic and we concentrate here on the situation where P is an overdetermined elliptic operator, i.e. $l > k$, though the case $l = k$ is also formally permitted. In fact, we need real analyticity of the coefficients of P in order to have information about solvability of the system $Pu = f$, or, in other words, about the validity of the Poincarè Lemma for the compatibility complex $\{E^i, P^i\}$ induced by P (see [T5], [AnNa]). The validity of the Poincarè Lemma for operators with smooth coefficients is an open problem and it will be discussed in Chapter 3.

What new facts does this bring to the Cauchy problem?

First, the differential operator P may have no right fundamental solutions. Hence the Green integral $\mathcal{G}\widetilde{u}$ (see (2.4.1)) may, perhaps, not satisfy the equation $P\mathcal{G}\widetilde{u} = 0$.

On the other hand, every overdetermined differential operator P induces on the hypersurface S a tangential differential operator P_b, and now "the initial data" $(\oplus u_j)$ must satisfy the induced tangential equation on S (see Tarkhanov [T5], §11). We denote by $\{C_j\}_{j=0}^{p-1}$ the Dirichlet system of order $(p - 1)$ on ∂D associated to the system $\{B_j\}$ in the Green formula for the differential operator P. This system is determined in a natural way in Lemma 1.1.6.

LEMMA 2.10.2. *If Problem 2.10.1 is solvable then* $P_b(\oplus u_j) = 0$ *(weakly) on* S, *that is,*

(2.10.1)
$$\int_S < C_j(P^{1'} v), u_j >_y ds = 0 \quad for\, all\, v \in \mathcal{D}(E^{2'})\, such\, that\, (supp\, v) \cap \partial D \subset S.$$

PROOF. Let there be a solution $u \in S^f(D)$ such that $B_j f = u_j$ $(0 \le j \le p-1)$ on S. Then, if $v \in \mathcal{D}(E^{2'})$ and $(supp\, v) \cap \partial D \subset S$, the Stokes formula implies

$$\int_S < C_j((P^1)'v), u_j >_y ds = \int_{\partial D} < C_j((P^1)'v), B_j f >_y ds =$$

$$= \lim_{\varepsilon \to +0} \int_{\partial D_\varepsilon} G_P((P^1)'v), f) = 0,$$

which was to be proved. $\square$

Let $O \Subset X$ be a domain and S be a smooth closed hypersurface in O dividing this domain into two connected components: $O^- = D$ and $O^+ = O \backslash \overline{D}$. For our purposes, it is sufficient to consider that the Dirichlet system $\{B_j\}$ is given only in some neighbourhood of (compact) S.

We had already noted in §1.1 that, the differential operator $\Delta = P^*P$ has a (bilateral) fundamental solution $\Phi \in pdo_{-2p}(E \to E)$ whose kernel is real analytic off the diagonal of $X \times X$ (see Tarkhanov [T5], §8).

We consider the following system of boundary operators defined in the neighbourhood U of the boundary ∂D. For a section $u \in C^{p-1}_{loc}(E_{|U})$ we set $\tau(u) = \oplus(B_j u)$, that is, $\tau(u)$ is a representation of the Cauchy data on S with respect to the differential operator P. Similarly for $g \in C^{p-1}_{loc}(F_{|U})$ we set $\nu(g) = \oplus(*^{-1} C_j * g)$, that is, $\nu(g)$ represents the Cauchy data of g on S with respect to the differential operator P^*.

LEMMA 2.10.3. *The system of boundary operators* $\{\tau(.), \nu(P.)\}$ *forms a Dirichlet system of order* $(2p-1)$ *on* ∂D.

PROOF. This fact has already been noted in the proof of Theorem 1.4.4, and it is proved by simple calculations. $\square$

For easy reference we note a simple consequence of Theorem 1.3.6.

LEMMA 2.10.4. *Let* $S \in C^\infty_{loc}$. *Then, for any solution* $u \in S^f(O^\pm)$ *which has finite order of growth near* S, *the expressions* $\tau(u)$ *and* $\nu(Pu)$ *have weak limit values on* S *belonging to* $\mathcal{D}'(\oplus F_{j|S})$.

PROOF. The statement of the lemma follows from Theorem 1.3.6 and Lemma 2.10.3 because, for any domain $D' \subset O^\pm$ whose boundary intersects the boundary of $O^\pm$ only in the set S, the restriction of the solution u on D' belongs to $S^f_\Delta(D')$, and because it is possible to extend the Dirichlet system $\{\tau(.), \nu(P.)\}$ from $\partial D' \cap S$ to the whole boundary $\partial D'$ as a suitable Dirichlet system (at least, if the boundary of $\partial D'$ is sufficiently smooth). $\square$

We could not prove the converse statement (as we did in Theorem 1.3.6) except in the case when S is a connected component of the boundary of the domain $O^\pm$.

LEMMA 2.10.5. *Let $S \in C^\infty_{loc}$. If the solutions $u^\pm \in S_\Delta(O^\pm)$ have finite orders of growth near S, and $\tau(u^+) = \tau(u^-)$ and $\nu(Pu^+) = \nu(Pu^-)$ on S then there is a solution $u \in S_\Delta(O)$ such that $u_{|O^\pm}) = u^\pm$.*

PROOF. It is sufficient to use Theorem 3.2 from the book of Tarkhanov [T4] taking into consideration Lemma 2.10.3. □

The following theorem for the Cauchy-Riemann system in the space $\mathbb{C}^n$ was first proved, apparently, by Kytmanov (see Aizenberg and Kytmanov [AKy]).

THEOREM 2.10.6. *We suppose that $S \in C^\infty_{loc}$. If a solution $u \in S_\Delta(D)$ has finite order of growth near S, and $P_b(\tau(u)) = 0$, and $\nu(Pu) = 0$ on S then $Pu = 0$ everywhere in the domain D.*

PROOF. Let the solution $u \in S_\Delta(D)$ have finite order of growth near the hypersurface S. Then, from Lemma 2.10.4, the expressions $\tau(u)$ and $\nu(Pu)$ have weak limit values on S belonging to $\mathcal{D}'(\oplus F_{j|S})$. We suppose that $P_b(\tau(u)) = 0$, and $\nu(Pu) = 0$ on S.

Fix an arbitrary point $x^0 \in S$. Since the coefficients of the differential operator P are real analytic and P has an injective symbol then the complex of compatibility conditions $\{E^i, P^i\}$ (which is induced by P) is exact in positive degrees on the level of sheaves over X. In particular, this means that for any neighbourhood $U = U(x^0)$ of the point x^0 and any section $f \in S_{P^1}(U)$ there exist a possibly smaller neighbourhood $V = V(x^0)$ of this point, and a section $v \in C^\infty_{loc}(E_{|V})$ such that $Pv = f$ on V (see Tarkhanov [T5], Theorem 3.10).

Since $\tau(u)$ represents the Cauchy data of u on S with respect to the differential operator P, and $P_b(\tau(u)) = 0$ on S then the exact Mayer -Vietoris sequence (see Theorem 18.9 in the book of Tarkhanov [T5]) implies that there are a neighbourhood $V = V(x^0)$ of the point x^0 in O and solutions $u^\pm \in S_\Delta(O^\pm \cap V)$ having finite order of growth near $S \cap V$ such that $\tau(u^+) - \tau(u^-) = \tau(u)$ on $S \cap V$.

Consider now two sections $\mathcal{F}^+ = u^+$ and $\mathcal{F}^- = u^- + u$ defined on the open sets $O^+ \cap V$ and $O^- \cap V$ respectively.

By construction, the sections $\mathcal{F}^\pm \in S_\Delta(O^\pm \cap V)$ have finite orders of growth near the hypersurface $S \cap V$, and $\tau(\mathcal{F}^+) = \tau(\mathcal{F}^-)$, and $\nu(P\mathcal{F}^+) = 0 = \nu(P\mathcal{F}^-)$ on $S \cap V$. Hence we can use Lemma 2.10.5, and conclude that there exists a section $\mathcal{F} \in S_\Delta(V)$ such that $\mathcal{F}_{|O^\pm \cap V} = \mathcal{F}^\pm$.

The differential operator Δ is elliptic and has real analytic coefficients therefore the theorem of Petrovskii implies that the sections $\mathcal{F}$ and $P\mathcal{F}$ are real analytic in V. Since $P\mathcal{F} = 0$ in $O^+ \cap V$, we can conclude that $P\mathcal{F} = 0$ everywhere in V.

Thus, $Pu = P\mathcal{F} - P\mathcal{F}^- = 0$ in $D \cap V$, and u is real analytic in the domain D. Hence we have $Pu = 0$ everywhere in this domain which was to be proved. □

We note that without the requirement "$P_b(\tau(u)) = 0$ on S" Theorem 2.10.6 is false.

EXAMPLE 2.10.7. Let $P(D) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ be the gradient operator in $\mathbb{R}^n$ $(n > 1)$, and $B_0 = 1$. Then $\Delta = P^*P$ is (minus) the usual Laplace operator in $\mathbb{R}^n$, and $\tau(u) = u$, and $\nu(Pu) = \frac{\partial u}{\partial \nu}$. In particular, if S is a piece of the hypersurface $\{x_n = 0\}$, any harmonic function u in D which does not depend on the variable x_n

satisfies $\nu(Pu) = 0$ on S. But, certainly, such a function may be non-constant in D. □

At the same time, if $S = \partial D$ then the condition "$P_b(\tau(u)) = 0$ on an open subset of S" in Theorem 10.3 is not necessary (see Karepov and Tarkhanov [KT2]).

Remark 2.10.8. As one can see from the proof of Theorem 1.3.6, the smoothness condition for the hypersurface S in Lemmata 2.10.3, 2.10.4, 2.10.5 and Theorem 2.10.6 can be loosened if we consider à priori solutions of the system $Pu = 0$ of order of growth which is not greater than a given fixed number. But this is a general observation. □

Theorem 2.10.6 gives a method of studying Problem 2.10.1. More precisely it shows that this problem is equivalent to the Cauchy problem for solutions of the system $P^*Pu = 0$ with initial data $\tau(u) = \oplus u_j$ and $\nu(Pu) = 0$ on S. The last problem belongs already to the range of Cauchy problems for determined elliptic systems which was considered in §2.3-2.9 of this chapter.

In the following sections we realize this method. æ

§2.11. Green's integral and solvability of the Cauchy problem for systems with injective symbols

We formulate Problem 2.10.1 more precisely (as we did in §2.4).

PROBLEM 2.11.1. *Let $u_j \in B^{s-b_j-1/q,q}(F_{j|\overline{S}})$ $(0 \le j \le p-1)$ be known sections on S where $s \in \mathbb{Z}_+$, and $1 < q < \infty$. It is required to find a section $u \in S_P(D) \cap W^{s,q}(E_{|D})$ such that $B_j u = u_j$ $(0 \le j \le p-1)$ on S.*

Using the "initial" data of Problem 10.1 we construct the Green integral in a special way.

Namely, as a left fundamental solution of the differential operator P we take the kernel $\mathcal{L}(x, y) = P^{*'}\Phi(x, y)$ where Φ is a fundamental solution of the "laplacian" $\Delta = P^*P$ about which we spoke in Lemma 2.10.1.

We denote by $\widetilde{u} \in B^{s-b_j-1/q,q}(F_{j|\partial D})$ $(0 \le j \le p-1)$ an extension of the section u_j to the whole boundary. If, for example, $s = 0$ and $u_j \in L^2(F_{j|S})$ $(0 \le j \le p-1)$, it is possible to extend them by zero on $\partial D \backslash S$. In any case the extensions could be chosen so that they will be supported on a given neighbourhood of the compact S on ∂D. Then we set $\widetilde{u} = \oplus u_j$, and

$$\mathcal{G}(\widetilde{u})(x) = -\int_{\partial D} < C_j \mathcal{L}(x, .), \widetilde{u_j} >_y ds \quad (x \in \partial D) \tag{2.11.1}$$

LEMMA 2.11.2. *The potential $\mathcal{G}(\widetilde{u})$ satisfies $\Delta\mathcal{G}(\widetilde{u}) = 0$ on each of the open sets D and $X \backslash \partial D$, and has finite order of growth near the surface ∂D.*

PROOF. This follows from equality (2.11.1) and the structure of the fundamental solution $\mathcal{L}(x, y)$. □

In particular, if we denote by $\mathcal{F}^{\pm}$ the restrictions of the section $\mathcal{F} \in \mathcal{D}'(E_{|O})$ to the sets $O^{\pm}$, we have $\mathcal{G}(\widetilde{u})^{\pm} \in S_\Delta(O^{\pm})$.

THEOREM 2.11.3. *If the boundary of the domain D is sufficiently smooth then, for Problem 2.11.1 to be solvable, it is necessary and sufficient that*

1. *the integral $\mathcal{G}(\widetilde{u})$ extends from O^+ to the whole domain O as a solution belonging to $S_\Delta(O) \cap W^{s,q}(E_{|O})$;*
2. *$P_b(sf) = 0$ in a neighbourhood of some point x^0 on S.*

PROOF. Necessity. Suppose that there is a section $u \in S_P(D) \cap W^{s,q}(E_{|D})$ such that $B_j u = u_j$ $(0 \le j \le p-1)$ on S.

We consider in the domain O (more exactly, in $O \backslash S$) the following section:

$$\mathcal{F}(x) = \begin{cases} \mathcal{G}\widetilde{u}(x), x \in O^+, \\ \mathcal{G}\widetilde{u}(x) - u(x), x \in O^-. \end{cases} \tag{2.11.2}$$

Using the boundedness theorem for potential operators in Sobolev spaces on manifolds with boundary (see Rempel and Schulze [ReSz], 2.3.2.5) we can conclude that $\mathcal{G}(\widetilde{u})^{\pm} \in W^{s,q}(E_{|O^{\pm}})$ (if the surface ∂D is sufficiently smooth, for example if $\partial D \in C^r$, $r = max(s, p-s)$). This means that $\mathcal{F}^{\pm} \in W^{s,q}(E_{|O^{\pm}})$.

On the other hand, we consider the difference $\delta = \mathcal{G}(\widetilde{u}) - \mathcal{G}(\oplus B_j u)$. Let $\varphi_\varepsilon \in D(X)$ be any function supported on the ε-neighbourhood of the set $\partial D \backslash S$, and being equal to 1 in some smaller neighbourhood of this set. Since $B_j u = \widetilde{u}_j$ $(0 \le j \le p-1)$ on S then we can write

$$\delta(x) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathcal{L}(x,.), \varphi_\varepsilon (B_j u - \widetilde{u}_j) >_y ds (x \notin \partial D).$$

The right hand side of this equality is a solution of the system $\Delta f = 0$ everywhere in the domain O except the part of the ε-neighbourhood of the boundary of S on ∂D which belongs to O. Therefore, since $\varepsilon > 0$ is arbitrary, $\delta \in S_\Delta(O)$.

Now expressing the integral $\mathcal{G}(\oplus B_j u)$ from the Green formula (1.3.4) and putting $\mathcal{G}(\widetilde{u}) = \mathcal{G}(\oplus B_j \widetilde{u}) + \delta$ in inequality (2.11.2) we obtain

$$\mathcal{F}(x) = \delta(x) \quad (x \in O \backslash S)$$

Hence the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $\Delta u = 0$.

Thus $\mathcal{F}$ belongs to $S_\Delta(O) \cap W^{s,q}(E_{|O})$, and on O^+ this section coincides with $\mathcal{G}(\widetilde{u})^+$, which was to be proved.

Sufficiency. Conversely, let $\mathcal{F} \in S_\Delta(O) \cap W^{s,q}(E_{|O})$ be a solution coinciding with $\mathcal{G}(\widetilde{u})^+$ on O^+, and $P_b(\oplus u_j) = 0$ in a neighbourhood of some point x^0 on S.

We set $u(x) = \mathcal{G}(\widetilde{u}) - \mathcal{F}(x)$ $(x \in D)$. The above mentioned boundedness theorem for potential operators in Sobolev spaces (see Rempel and Schulze [ReSz], 2.3.2.5) implies that $\mathcal{G}(\widetilde{u}) \in W^{s,q}(E_{|O^-})$. Therefore $u \in S_\Delta(D) \cap W^{s,q}(E_{|D})$, and u has finite order of growth near the hypersurface S.

Now Lemma 1.3.7 on the weak jump of the Green integral associated with the differential operator Δ and the Dirichlet system $\{\tau(.), \nu(P.)\}$ on ∂D implies that

$$\begin{cases} \tau(\mathcal{G}\widetilde{u}(x)^-) - \tau(\mathcal{G}\widetilde{u}(x)^+) = \oplus \widetilde{u}_j \, on \, \partial D, \\ \nu(P\mathcal{G}(\widetilde{u})^-) - \nu(P\mathcal{G}(\widetilde{u})^+) = 0 \, on \, \partial D. \end{cases}$$

Since $\tau(\mathcal{G}(\widetilde{u})^+) = \tau(\mathcal{F})$, and $\nu(P\mathcal{G}(\widetilde{u})^+) = \nu(P\mathcal{F})$ on S then these equations imply that

$$\begin{cases} \tau(u) = \oplus \widetilde{u}_j \ on \ S, \\ \nu(Pu) = 0 \ on \ S. \end{cases}$$

We use now the condition "$P_b(\oplus u_j) = 0$ in a neighbourhood $V = V(x^0)$ on S". Then $P_b(\tau(u)) = 0$ in V, and, from Theorem 2.10.6 applied to the piece $V \cap S$ instead of S, we obtain that $Pu = 0$ everywhere in the domain D.

Hence $u \in S_P(O) \cap W^{s,q}(E_{|O})$ is the required solution of Problem 2.11.1, which was to be proved. $\square$

For the Cauchy-Riemann operator in $\mathbb{C}^n$ $(n > 1)$ Theorem 2.11.3 is due to Aizenberg and Kytmanov [AKy].

There is an example showing that the sufficiency part of Theorem 2.11.3 without the requirement "$P_b(\oplus \widetilde{u}_j) = 0$ on an open subset of S" is false.

EXAMPLE 2.11.4. Let $P(D) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ be the gradient operator in $\mathbb{R}^n$ $(n > 1)$, and $B_0 = 1$. Then, as we note in Example 2.10.6, $\Delta = P^*P$ is (minus) the usual Laplace operator in $\mathbb{R}^n$, and $\tau(u) = u$, and $\nu(Pu) = \frac{\partial f}{\partial \nu}$. We take as S a piece of the hypersurface $\{x_n = 0\}$, and fix, on a neighbourhood of O, some non-constant harmonic function u which does not depend on the variable x_n. If the Cauchy data on S are given by means of the restriction $u_{|S}$ then the Green integral can be constructed by the formula $\mathcal{G}(\widetilde{u})(x) = \int_S \frac{\partial}{\partial \nu} g(x - .) f ds$, where $g(x-y)$ is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$. In other words, $\mathcal{G}(\widetilde{u})$ is (minus) the potential of a double layer with density f supported on S. From the theorems on the jump of this integral and its normal derivate, we have $\mathcal{G}(\widetilde{u})^- - \mathcal{G}(\widetilde{u})^+ = f$, and $\frac{\partial}{\partial \nu}\mathcal{G}(\widetilde{u})^- - \frac{\partial}{\partial \nu}\mathcal{G}(\widetilde{u})^+ = 0$ on S. Moreover $\frac{\partial f}{\partial \nu} = 0$ on S. Therefore Lemma 2.10.5 implies that the function $(\mathcal{G}(\widetilde{u}) - f)$ extends harmonically from O^+ to the whole domain O (by means of $\mathcal{G}(\widetilde{u})^-$ on O^-). This means that we can conclude the same for the integral $\mathcal{G}(\widetilde{u})^+$. However $u_{|S}$ may be the restriction of a non-constant function in D. $\square$

At the same time, if $S = \partial D$ then the condition "$P_b(\oplus \widetilde{u}_j) = 0$ on an open subset of S" in Theorem 2.11.3 is not necessary (see Karepov and Tarkhanov [KT2]).

COROLLARY 2.11.5 (THE CARTAN-KÄHLER THEOREM). *Suppose that the hypersurface S, the coefficients of the operators B_j $(0 \le j \le p-1)$ in a neighbourhood of ∂D and the sections $u_j \in D'(F_{j|S})$ $(0 \le j \le p-1)$ are real analytic. Then, if $P_b(\oplus u_j) = 0$ on S, there is a section u satisfying $Pu = 0$ in some neighbourhood of S and such that $B_j u = u_j$ $(0 \le j \le p-1)$ on S.*

PROOF. In view of the uniqueness theorem for solutions of $Pu = 0$ it is sufficient to find for each point $x^0 \in S$ a neighbourhood $V = V(x^0)$ on X and a solution $u \in S(V)$ such that $B_j u = u_j$ $(0 \le j \le p-1)$ on $S \cap V$. Therefore we can at once consider that the sections u_j $(0 \le j \le p-1)$ are real analytic in a neighbourhood of the compact S. Then we can construct the Green integral by the formula

$$\mathcal{G}(\widetilde{u})(x) = -\int_S < C_j \mathcal{L}(x, .), u_j >_y ds \quad (x \notin S).$$

The condition of the corollary implies that the integral $\mathcal{G}(\widetilde{u})$ is a real analytic (vector-) function up to S on each sides of this hypersurface. This means that each of the integrals $\mathcal{G}(\widetilde{u}^{\pm})$ extends as a solution of the system $\Delta f = 0$ to some neighbourhood of S. If we keep the same notations for these extensions then the difference $u = \mathcal{G}(\widetilde{u})^- - \mathcal{G}(\widetilde{u})^+$ is the solution we sought. $\square$

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§2.12. A solvability criterion for the Cauchy problem for systems with injective symbols in the language of space bases with double orthogonality

Theorem 2.11.3 has been formulated so that the application of the theory of §2.1 is suggested. For this assume in addition that $q = 2$.

So, in this section we consider the solvability aspect of Problem 2.11.1.

PROBLEM 2.12.1. *Under what conditions on the sections* $u_j \in W^{s-b_j-1/2,2}(F_{j|\overline{S}})$ $(0 \le j \le p-1)$ *is there a solution* $u \in S_P(D) \cap W^{s,2}(E_{|D})$ *such that* $B_j u = u_j$ $(0 \le j \le p-1)$ *on* S *?*

Let Ω be some relatively compact subdomain of O^+.

Since $\Omega \Subset O^+$, the restriction to Ω of the Green integral $\mathcal{G}(\widetilde{u})$ defined by equality (2.11.1) belongs to the space $S_\Delta(\Omega) \cap W^{s,2}(E_{|\Omega})$. Hence the extendibility condition for $\mathcal{G}(\widetilde{u})$ from O^+ to the whole domain O (as a solution in the class $S_\Delta(O) \cap W^{s,2}(E_{|O})$ could be obtained by the use of a suitable system $\{b_\nu\}$ in $S_\Delta(O) \cap W^{s,2}(E_{|O})$ with the double orthogonality property. More exactly, it is required that $\{b_\nu\}$ should be an orthonormal basis in $\Sigma_1 = S_\Delta(O) \cap W^{s,2}(E_{|O})$ and an orthogonal basis in $\Sigma_2 = S_\Delta(\Omega) \cap W^{s,2}(E_{|\Omega})$.

Since $\Delta = P^*P$ is an elliptic differential operator with real analytic coefficients on X, Theorem 2.5.5 guarantees existence of such a basis $\{b_\nu\}$, at least if the boundary of Ω is regular (see §2.5). As we did in §2.5, for an element $\mathcal{F} \in \Sigma_1$ we shall denote by $c_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthonormal system $\{b_\nu\}$ in Σ_1, that is, $c_\nu(\mathcal{F}) = (\mathcal{F}, b_\nu)_{H_1}$. And for an element $\mathcal{F} \in \Sigma_2$ we shall denote by $k_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthogonal system $\{Tb_\nu\}$ in Σ_2, that is, $k_\nu(\mathcal{F}) = \frac{(\mathcal{F}, Tb_\nu)_{H_2}}{(Tb_\nu, Tb_\nu)_{H_2}}$.

We formulate now the solvability conditions for Problem 2.12.1. Let $\mathcal{G}\widetilde{u}$ be the Green integral (see (2.11.1) constructed with "initial" data of the problem. As we noted, the restriction of the section $\mathcal{G}\widetilde{u}$ to Ω belongs to the space Σ_2.

LEMMA 2.12.2. *For* $\nu = 1, 2, ...$

$$k_\nu(\mathcal{G}\widetilde{u}) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\mathcal{L}(., y)), \widetilde{u}_j >_y ds. \tag{2.12.1}$$

PROOF. This consists of direct calculations with the use of equality (2.11.1). □

In order to determine the coefficients $k_\nu(\mathcal{G}\widetilde{u})$ $(\nu = 1, 2, ...)$ it is not necessary to know the basis $\{Tb_\nu\}$ in Σ_2. It is sufficient only to know the coefficients of the decomposition of the fundamental matrix $(\mathcal{L}(., y)$ $(y \in \partial D)$ with respect to this series. The properties of the coefficients $k_\nu(\mathcal{L}(., y) \in C^\infty_{loc}(F^*_{|X\setminus\Omega})$ we shall discuss in §2.13.

THEOREM 2.12.3. *If the boundary of the domain D is sufficiently smooth then for the solvability of Problem 2.12.1 it is necessary and sufficient that*

(1) $\sum_{\nu=1}^\infty |k_\nu(\mathcal{G}\widetilde{u})|^2 < \infty$;
(2) $P_b(\oplus u_j) = 0$ *in a neighborhood of some point* x^0 *on* S.

PROOF. The statement follows from Theorem 2.11.3 as Theorem 2.5.8 follows from Theorem 2.4.2. $\square$

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§2.13. Carleman's formula

In this section we consider the regularization aspect of Problem 2.11.1.

PROBLEM 2.13.1. *It is required to find a solution* $f \in S_P(D) \cap W^{s,2}(E_{|D})$ *using known values* $B_j u \in W^{s-b_j-1/2,2}(F_{j|\overline{S}})$ $(0 \leq j \leq p-1)$ *on* S.

It is easy to see from Corollary 2.1.9 that side by side the solvability conditions for Problem 2.4.1 ($q = 2$) bases with double orthogonality give the possibility to obtain a suitable formula (of Carleman) for the regularization of solutions. We shall illustrate this on example of Problem 2.6.1.

Let $\{b_\nu\}$ be the basis with double orthogonality, used in the previous section, in the space $(\Sigma_1 =) S_P(O) \cap W^{s,2}(E_{|O})$ such that the restriction of $\{b_\nu\}$ to Ω (that is, $\{Tb_\nu\}$) is an orthogonal basis of $(\Sigma_2 =) S_P(\Omega) \cap W^{s,2}(E_{|\Omega})$.

As above, we denote by $\{k_\nu(\mathcal{L}(.,y))\}$ the sequence of Fourier coefficients for the fundamental matrix $\mathcal{L}(.,y)$ $(y \in \Omega)$ with respect to the system $\{Tb_\nu\}$, i.e.,

$$k_\nu(\mathcal{L}(.,y)) = \frac{1}{\lambda_\nu} \int_\Omega < *D^\alpha b_\nu, D^\alpha \mathcal{L}(.,y) >_y dv \quad (\nu = 1,2...). \tag{2.13.1}$$

LEMMA 2.13.2. *The sections* $k_\nu(\mathcal{L}(.,y))$ $(\nu = 1,2...)$ *are continuous, together with their derivatives up to order* $(p-s-1)$, *on the whole set* X.

PROOF. See, Lemma 2.6.2. $\square$

Using formula (2.13.1) one can see that the sections $k_\nu(\mathcal{L}(.,y))$ $(\nu = 1,2...)$ extend to the boundary of Ω from each side as infinitely differentiable sections (at least, if the boundary is smooth).

LEMMA 2.13.3. *For any number* $\nu = 1,2,...$ *we have* $P' k_\nu(\mathcal{L}(.,y)) = 0$ *everywhere in* $X \backslash \overline{\Omega}$.

PROOF. See Lemma 2.6.3. $\square$

We consider the following kernels $\mathfrak{C}^{(N)}(x,y)$ defined for $(x,y) \in O \times X$ $(x \neq y)$:

$$\mathfrak{C}^{(N)}(x,y) = \mathcal{L}(x,y) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\mathcal{L}(.,y)) \quad (N = 1,2,...). \tag{2.13.2}$$

LEMMA 2.13.4. *For any number* $N = 1,2,...$ *the kernels* $\mathfrak{C}^{(N)} \in C_{loc}(E \boxtimes F)$ *satisfy* $P(x)\mathfrak{C}^{(N)}(x,y) = 0$ *for* $x \in O$, *and* $P'(y)\mathfrak{C}^{(N)}(x,y) = 0$ *for* $y \in X \backslash \Omega$ *everywhere except the diagonal* $\{x = y\}$.

PROOF. Since $\{b_\nu\} \subset S_\Delta(O)$, this immediately follows from Lemma 2.13.3. $\square$

From the following lemma one can see that the sequence of kernels $\{\mathfrak{C}^{(N)}\}$, suitably, for example in a piece-constant way, interpolated to real values $N \geq 0$, provides a special Carleman function for Problem 2.13.1 (see Tarkhanov [T4], §25).

LEMMA 2.13.5. *For any multi-index* α, $D_y^\alpha \mathfrak{C}^{(N)}(.,y) \to 0$ *in the norm of* $W^{s,2}(E \otimes F^*_{y|O})$ *uniformly with respect to* y *on compact subsets of* $X \setminus \overline{O}$, *and even* $X \setminus O$ *if* $|\alpha| < p - s - n/2$.

PROOF. See Lemma 2.6.6. □

We can formulate now the main result of the section. For $u \in S_P(D) \cap W^{s,2}(E_{|D}))$ we denote by $\widetilde{u} \in W^{s-b_j-1/2,2}(G_{j|\partial D})$ $(0 \le j \le p-1)$ an (arbitrary) extension of the section $B_j u$ from S to the whole boundary.

THEOREM 2.13.6 (CARLEMAN'S FORMULA). *For any solution* $u \in S_P(D) \cap W^{s,2}(E_{|D})$ *the following formula holds:*

$$u(x) = -\lim_{N\to\infty} \int_{\partial D} < C_j \mathfrak{C}^{(N)}(x,.), \widetilde{u}_j >_y ds \quad (x \in D). \tag{2.13.3}$$

PROOF. This follows from Theorems 2.13.3 and 2.12.8 as Theorem 2.6.7 follows from Theorems 2.4.2 and 2.5.8. □

We emphasize that the integral on the right hand side of formula (2.13.3) depends only on the values of the expressions $B_j u$ $(0 \le j \le p-1)$ on S. Thus this formula is a quantitative expression of (uniqueness) Theorem 2.2.2. However this gives much more than the uniqueness theorem because there is sufficiently complete information about the Carleman function $\mathfrak{C}^{(N)}$.

For holomorphic functions of several variables the Carleman formula (2.13.3) is first met, apparently, in [ShT4].

Remark 2.13.7. The series $\sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{u}) b_\nu$ (defining the solution $\mathcal{F}$) converges in the norm of the space $W^{s,2}(E_{|O})$. The Stieltjes-Vitali theorem (see Hörmander [Hö2], 4.4.2) implies now that it converges together with all its derivatives on compact subsets of O. Then, as in §2.6, one can see that the limit in (2.13.3) is reached in the topology of the space $C^\infty_{loc}(E_{|O})$.

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§2.14. Examples for matrix factorizations of the Laplace operator

2.14.1. The Cauchy problem for matrix factorizations of the Laplace operator.

The examples of this section are based on the following simple observation.

LEMMA 2.14.1.1. *If the coefficients of the differential operator* P *are real analytic then Problem 2.11.1 is solvable if and only if*

(1) *the section* $\mathcal{G}(\widetilde{u})$ *extends from* O^+ *to the whole domain* O *as a real analytic section belonging to* $W^{s,q}(E_{|D})$;
(2) $P_b u_0 = 0$ *in a neighbourhood of some point* $x^0 \in S$.

PROOF. First, we note that, since $P\mathcal{G}(\widetilde{u}) = 0$ outside of ∂D, the section $\mathcal{G}(\widetilde{u})$ is real analytic in the domain O^+. Now let $\mathcal{F}$ be the above extension of this section in O. Then $P\mathcal{F}$ is also a real analytic section in O, and $P\mathcal{F} = 0$ in O^+. From the uniqueness theorem we obtain that $P\mathcal{F} = 0$ everywhere in the domain O, that is, $\mathcal{F} \in S_P(O) \cap W^{s,2}(E_{|O})$. Therefore the statement of the lemma follows from Theorem 2.11.3. □

In particular, we can use the fact that $(P^*P)\mathcal{G}\widetilde{u} = 0$ everywhere outside ∂D, and the extendibility condition for $\mathcal{G}(\widetilde{u})$ (up to a section $\mathcal{F} \in W^{s,2}(E_{|O})$ satisfying $(P^*P)\mathcal{F} = 0$ in O) write in the language of bases with the double orthogonality.

DEFINITION 2.14.1.2. The differential operator P is said to be a matrix factorization of the Laplace operator if $p = 1$, and $P^*P = -\Delta_n I_k$ where Δ_n is the Laplace operator in $\mathbb{R}^n$.

PROBLEM 2.14.1.3. *Let $u_0 \in C_{loc}(E_{|S})$ be a summable section of E on S. It is required to find a solution $u \in S_P(D) \cap C_{loc}(E_{|D\cup S})$ such that $u_{|S} = u_0$.*

As the fundamental solution of the differential operator P we can take the matrix $\mathcal{L}(x,y) = P^{*'}(y)\varphi_n(x-y)$, where $\varphi_n(x-y)$ is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$ with the opposite sign. Then the Green integral (2.11.1) has the following form:

$$\mathcal{G}\widetilde{u}(x) = -\frac{1}{\sqrt{-1}} \int_S {}^t\mathcal{L}(x,y) \sum_{\alpha=1} P_\alpha(y)\nu^\alpha(y)u_0(y)ds(y) \ (x \notin S),$$

where $\nu(y)$ is the vector of unit outward normal to S at the point y.

It is easy to see from the structure of the fundamental matrix $\mathcal{L}$ that the components of the section $\mathcal{G}\widetilde{u}$ are harmonic functions everywhere in B_R (and even in $\mathbb{R}^n$) except on the set S.

In the next 2 subsections we suppose that P is differential operator as in Definition 2.14.1.2.

2.14.2. Example for matrix factorization of the Laplace operator in a part of a ball in $\mathbb{R}^n$.

Let $O = B_R$ be the ball in $\mathbb{R}^n$ with centre at zero and of radius $0 < R < \infty$, and S be a smooth closed hypersurface in B_R dividing this ball into 2 connected components O^+, and $D = O^-$ so that the domain O^+ contains zero. We consider the following problem (of Cauchy).

To obtain a solvability criterion for Problem 2.14.1.1 we can use the basis with double orthogonality constructed in Lemma 2.8.2.2.

We fix $0 < r < dist(0,S)$ and set $\Omega = B_r$ so that $\Omega \Subset O$. In order to obtain the Fourier coefficients for the section $\mathcal{G}(\widetilde{u})$ with respect to this basis in $h^2(B_r)$ it is sufficient to know the Fourier coefficients for the fundamental matrix $\mathcal{L}(x,y)$ (see (2.12.1)). The information about them is contained in the following lemma.

LEMMA 2.14.2.1.

$$\mathcal{L}(x,y) = \mathcal{L}(0,y) - \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} h_\nu^{(i)}(x)P^{*'}(y)\left(\frac{1}{n+2\nu-2}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}\right). \tag{2.14.2.1}$$

where the series converges together with all the derivatives uniformly on compact subsets of the cone $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

PROOF. It is sufficient to use the similar decomposition for $\varphi_n(x-y)$ obtained in Lemma 2.8.2.1. □

Our principal result will be formulated in the language of the coefficients

$$k_\nu^{(i)} = \frac{1}{\sqrt{-1}}\int_S P^{*'}(y)\left(\frac{1}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}\right)\sigma(P)(\nu)u_0 ds \ (\nu = 1,2,...).$$

THEOREM 2.14.2.2. *For solvability of Problem 2.14.1.3, it is necessary and sufficient that*

(1) $\limsup_{\nu\to\infty} \max_i \sqrt[\nu]{|k_\nu^{(i)}(y)|} \leq \frac{1}{R}$;
(2) $P_b u_0 = 0$ *in a neighbourhood of some point* $x^0 \in S$.

PROOF. The statement follows from Lemma 2.14.1.1 as Theorem 2.8.2.4 follows from Theorem 2.8.1.3. $\square$

Remark 2.14.2.3. It is clear that if P is determined elliptic then $P_b \equiv 0$, i.e. condition (2) of Theorem 2.14.2.2 obviously holds.

Let us give now the corresponding variant of Carleman's formula. For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x,y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) + \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} h_\nu^{(i)}(x) P^{*'}(y) \left(\frac{1}{(n+2\nu-2)} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \right).$$

LEMMA 2.14.2.4. *For any number $N = 1, 2, ...,$ the kernel $\mathfrak{C}^{(N)}$ is an infinitely differentiable section of $E \boxtimes F$, harmonic with respect to x, and satisfying $P'(y)\mathfrak{C}^{(N)}(x,y) = 0$ for all $y \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. Follows from the properties of the matrix $\mathcal{L}$ and the polynomials $h_\nu^{(i)}(y)$. $\square$

We note that since $\mathfrak{C}^{(N)}$ is a "remainder" summand in the formula (2.14.1), $\mathfrak{C}^{N)}(x,y) \to 0$ $(N \to \infty)$, together with all its derivatives uniformly on compact subsets of the cone $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

THEOREM 2.14.2.5 (CARLEMAN'S TYPE FORMULA). *For any solution $u \in S_P(D) \cap C_{loc}(E_{|D\cup S})$ whose restriction to S is summable there, the following formula holds*

$$u(x) = -\frac{1}{\sqrt{-1}} \lim_{N\to\infty} \int_S \mathfrak{C}^{(N)}(x,.)\sigma(P)(\nu)u_0 ds \quad (x \in D). \tag{2.14.2.2}$$

PROOF. This is similar to the proof of Theorem 2.13.6. $\square$

Remark 2.14.2.6. As in Theorem 2.13.6, the convergence of the limit in (2.14.2.2) is uniform on compact subsets of the domain D together with all its derivatives.

EXAMPLE 2.14.2.7. Let $P = 2\frac{d}{dz}$ be the Cauchy-Riemann system in $\mathbb{C}^1$ ($\cong \mathbb{R}^2$). Obviously P is an determined matrix factorization of the Laplace operator in $\mathbb{R}^{2n}$. Then $P_b \equiv 0$, the corresponding fundamental solution is the Cauchy kernel $\mathcal{L}(\zeta, z) = \frac{-1}{\zeta - z}$ where $z = x_1 + \sqrt{-1}x_2$, $\zeta = y_1 + \sqrt{-1}y_2$, $x, y \in \mathbb{R}^2$, and the corresponding Green's integral is the Cauchy integral. The system of the monomials $\{1, z^\nu, \overline{z}^\nu\}_{\nu=1}^{\infty}$ is the basis with double orthogonality constructed in Lemma 2.8.1.5. The corresponding solvability conditions for the Cauchy problem were obtained by L. Aizenberg (see [AKy]). The corresponding Carleman's formula, probably, is due to Goluzin and Krylov (see [A]); it is one of the simplest formulae of this type. More exactly, for any holomorphic function $u \in C_{loc}(E_{|D\cup S})$ whose restriction to S is summable there the following formula holds:

$$u(z) = \lim_{N\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_S \left(\frac{z}{\zeta}\right)^N \frac{u_0(\zeta) d\zeta}{\zeta - z} \ (z \in D).$$

EXAMPLE 2.14.2.8. Let $P = 2\begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \dots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$ be the Cauchy-Riemann system in $\mathbb{C}^n$ ($\cong \mathbb{R}^{2n}$). Obviously P is an overdetermined matrix factorization of the Laplace operator in $\mathbb{R}^{2n}$. Then condition (2) in Theorem 2.14.5 is the well known tangential Cauchy-Riemann condition (or, CR-condition) on S. The corresponding Green's integral is the Martinelli-Bochner integral (see §1.2). Theorem 2.14.5 was proved in this case by Aizenberg and Kytmanov [AKy]. The corresponding Carleman's formula is due to Shlapunov and Tarkhanov [ShT4].

2.14.3. Example for matrix factorization of the Laplace operator in a shell in $\mathbb{R}^n$.

In this section we consider the Cauchy problem for $(k \times k)$-matrix factorizations if the Laplace operator in a shell D in $\mathbb{R}^n$ whose exterior surface is a smooth closed hypersurface S in $\mathbb{R}^n$ and interior surface is a sphere $\partial B_r = \{x \in \mathbb{R}^n : \; |x| = r\}$ with centre at zero and radius $0 < r < \infty$, with the Cauchy data on S.

As in §2.8.3, $G(r_1, r_2) = \{x \in \mathbb{R}^n : \; r_1 < |x| < r_2\}$ is a shell with $0 < r_1 < r_2 < \infty$, R is a real number such that $D \Subset O = G(r, R)$, $O^- = D$, $O^+ = G(r, R) \backslash \overline{D}$. Then $\mathcal{F}_P^{\pm} = \mathcal{F}_{P|G^{\pm}}$.

As in §2.8.3, using Lemma 2.8.2.1, we can write the Laurent series for solutions of matrix factorizations of the Laplace operator in a shell $G(r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$ (cf. [T4], Corollary 8.9).

PROPOSITION 2.14.3.1. *Every vector-function $u \in C^1(E_{|\overline{G(r_1,r_2)}})$, satisfying $Pu = 0$ in $G(r_1, r_2)$, can be expanded as follows:*

$$u(x) = \sum_{\nu=0}^{\infty} \sum_{i=1}^{J(\nu)} a_\nu^{(i)} h_\nu^{(i)}(x) + b_0 \varphi_n(x) + \sum_{\nu=1}^{\infty} \sum_{i=1}^{J(\nu)} b_\nu^{(i)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}, \tag{2.14.3.1}$$

where the series converge absolutely together with all the derivatives uniformly on compact subsets of $G(r_1, r_2)$ and the coefficients $a_\nu^{(i)}$, $b_\nu^{(i)}$ are uniquely defined by

$$a_\nu^{(i)} = \begin{cases} \frac{-1}{n+2\nu-2} \int_{|y|=r_2} \left({}^tP^*(y) \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \right) \sum_{|\alpha|=1} P_\alpha(y) \nu^\alpha(y) u(y) ds(y) \; (\nu = 1, 2, \dots), \\ \sqrt{\sigma_n} \int_{|y|=r_2} ({}^tP^*(y) \varphi_n(y)) \sum_{|\alpha|=1} P_\alpha(y) \nu^\alpha(y) u(y) ds(y) \; (\nu = 0). \end{cases}$$

$$b_\nu^{(i)} = \frac{-1}{n+2\nu-2} \int_{|y|=r_1} {}^tP^*(y) h_\nu^{(i)}(y) \sum_{|\alpha|=1} P_\alpha(y) \nu^\alpha(y) u(y) ds(y) \; (\nu = 1, 2, \dots).$$

Now, denoting by

$$c_\nu^{(i)} = \frac{-1}{n+2\nu-2} \int_S {}^tP^*(y) h_\nu^{(i)}(y) \sum_{|\alpha|=1} P_\alpha(y) \nu^\alpha(y) u_0(y) ds(y) \; (\nu = 1, 2, \dots).$$

and arguing as in Theorem 2.8.3.3 we obtain the following result.

THEOREM 2.14.3.2. *Let u_0, $u_1 \in L^1(E_{|S})$. Then, for Problem 2.14.1.3 to be solvable, it is necessary and sufficient that*

$$\limsup_{\nu \to \infty} \max_i \sqrt[\nu]{|c_\nu^{(i)}|} \le r$$

Let us write the corresponding Carleman's formula.

For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x,y) = {}^tP^*(y)\varphi_n(x-y) - \sum_{\nu=1}^{N}\sum_{i=1}^{J(\nu)} \frac{{}^tP^*(y)h_\nu^{(i)}(y)}{(n+2\nu-2)} \frac{\overline{h_\nu^{(i)}(x)}}{|x|^{n+2\nu-2}}.$$

PROPOSITION 2.14.3.3. *For any number $N = 1, 2, ...,$ the kernel $\mathfrak{C}_\Delta^{(N)}$ is harmonic with respect to x and satisfying ${}^tP^*(y)\mathfrak{C}^{(N)}(x, y) = 0$ for all $x \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. Follows from the properties of the $\varphi_n(x-y)$ and the polynomials $h_\nu^{(i)}(y)$. □

The kernels $\mathfrak{C}^{(N)}$ are useful to obtain formulae for solutions and approximate solutions of Problem 2.14.1.3.

THEOREM 2.14.3.4 (CARLEMAN'S TYPE FORMULA).. *For any function $u \in C(E_{|D\cup S})$, satisfying $Pu = 0$ in D, whose restriction to S is summable there, the following formula holds*

$$u(x) = \lim_{N\to\infty} \int_S \mathfrak{C}^{(N)}(x,y) \sum_{|\alpha|=1} P_\alpha(y)\nu^\alpha(y)u(y)ds(y) \ (x \in D). \tag{2.14.3.2}$$

Remark 2.14.3.5. The convergence of the limit in (2.14.3.2) is uniform on compact subsets of the domain D together with all its derivatives.

We note that in the case, where P is the Cauchy-Riemann system $\mathbb{C}^n$, $n > 1$, the Cauchy Problem 2 is the Cauchy Problem for a bounded domain $D \cup B_r$ with the Cauchy datum u_0 on the whole boundary S (because of the Hartogs Theorem on removability of compact singularities of holomorphic functions in $\mathbb{C}^n$, $n > 1$). In particular it shows that formula (2.14.3.2) may be trivial for $(l \times k)$-matrix factorizations of the Laplace operator, with $l > k$ (as, for instance in the case of the multi-dimensional Cauchy-Riemann system).

EXAMPLE 2.14.3.6. Let $P = \frac{d}{d\overline{z}}$ be the Cauchy-Riemann in the complex plane $\mathbb{C}^1$. Then the system $\{\frac{1}{\sqrt{2\pi}}, \frac{z^\nu}{\sqrt{2\pi}}, \frac{\overline{z}^\nu}{\sqrt{2\pi}}\}$ (with $z = x_1 + \sqrt{-1}x_2$, $\overline{z} = x_1 - \sqrt{-1}x_2$, $(x_1, x_2) \in \mathbb{R}^2$) is the system of spherical harmonics $\{h_\nu^{(i)}\}$ $(\nu = 0, 1, ..., i = 1, 2)$. In this case D is a shell between a closed smooth curve S and a circle B_r, $G(r, R)$ is a ring with centre at zero, ${}^tP^*(\zeta)g(\zeta - z) = \frac{1}{\pi}\frac{1}{\zeta - z}$ is the Cauchy kernel and

$$\mathcal{G}(u_0)(z) = \frac{1}{2\pi\sqrt{-1}} \int_S \frac{u_0(\zeta)d\zeta}{\zeta - z} \ (z \notin S)$$

is the Cauchy integral. Decomposition (2.1) is the Laurent series for holomorphic functions.

By simple calculations, we have

$$c_\nu^{(i)} = \frac{-1}{\sqrt{-2\pi}} \int_S u_0(\zeta)\zeta^{\nu-1}d\zeta \ (\nu = 1, 2, ...).$$

$$\mathfrak{C}^{(N)} = \frac{1}{\pi}\left(\frac{\zeta}{z}\right)^{N+1}\frac{1}{\zeta - z}\ (N \geq 0).$$

Thus, we obtain Carleman's formula for a holomorphic function $u \in C(D \cup S)$, whose restriction to S is summable there:

(2.14.3.3) $$u(z) = \frac{1}{2\pi\sqrt{-1}}\lim_{N\to\infty}\int_S \left(\frac{\zeta}{z}\right)^N \frac{u(\zeta)d\zeta}{\zeta - z}\ (z \in D).$$

Formula (2.3) is well-known in the case where S is a circle ∂B_ρ with $r < \rho < R$ (see, for example, [A]).

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CHAPTER III

ITERATIONS OF GREEN'S INTEGRALS AND THEIR APPLICATIONS TO ELLIPTIC DIFFERENTIAL COMPLEXES

§3.0. Introduction

The validity of the Poincaré lemma, i.e. local acyclicity, for complexes of linear partial differential operators with smooth coefficients is a long standing problem of the theory of overdetermined systems (see, for example, [T5], [AnNa]). The Poincaré lemma is valid for complexes of linear partial differential operators with constant coefficients which are obtained from Hilbert resolutions of modules of finite type over the ring of polynomials (see, for instance, [Pal] and [Mal1], [Mal2]) and for elliptic complexes of linear partial differential operators with real analytic coefficients satisfying suitable algebraic conditions (see [AnNa]). Local solvability is also known for determined elliptic systems and has been thoroughly investigated for scalar operators of the principal type. Trivial examples shows that some nondegeneracy assumption is necessary, but a famous example of H. Lewy showed that even a nondegenerate scalar linear partial differential operator of the first order with polinomial coefficients in $\mathbb{R}^3$ can be nonsurjective on the germs of smooth functions at any point of $\mathbb{R}^3$.

The Hans-Lewy example has been extended in several ways and simple examples have been found even in $\mathbb{R}^2$. An interpretation of the Hans-Lewy phenomenon was given by [Hö3] for the case of a scalar operator and by [Na1] for complexes of partial differential operators. These results essentially involve some asymptotic analysis related to a microlocalization of the operator (or the complex) near the bicharacteristic points.

It is a natural question to investigate whether in case there are no bicharacteristic points (i.e. for elliptic complexes) there is local acyclicity.

In this chapter we take up this problem. Elliptic complexes are characterized by the exactness of the complex obtained by the principal symbols of their operators at each nonzero cotangent vector. Although we are still not able to prove the Poincaré Lemma for elliptic complexes, we prove in this chapter a representation formula giving a solution of the equation $Pu = f$ for an operator P with injective symbol whenever a solution exists.

This representation involves the sum of a series whose terms are iterations of integro- differential operators, while solvability of $Pu = f$ is equivalent to the convergence of the series together with an orthogonality condition with respect to a harmonic space (the last one is a trivial necessary condition).

For the Dolbeault complex, these integro-differential operators are related to the Mar- tinelli-Bochner integral. In this case, results similar to ours were obtained by A.V. Romanov [Rom2].

In fact this approach is more fit to study the global solvability of the system $Pu = f$ in a domain $D \Subset X$, though in this way a problem about the (global) regularity of solutions of $Pu = f$ in D arises (cf. [Sh3], [Sh6]).

Although the example of the Cauchy-Riemann system shows that in general we should expect a loss of global Sobolev regularity for the solutions of $Pu = f$ (see Example 3.6.4), the case where the system can be solved without losing global regularity has interesting applications to a variational nonelliptic boundary value problem, that we discuss and illustrate by the examples at the end of the chapter.

Let us describe in a more precise way the contents of this chapter.

Let X be an open set in $\mathbb{R}^n$ $(n \geq 1)$ and P be an elliptic $(l \times k)$-matrix of partial differential operators of order $p \geq 1$ with C^∞ coefficients in X. We are interested in the solvability of the equation $Pu = f$ in a relatively compact domain D in X. Our approach is based on the following simple but useful observation.

Let H be a linear topological vector space of (vector-valued) functions defined in D and let us assume that for every $u \in H$ the following formula holds true:

$$u = \Pi_1 u + \Pi_2 P u \tag{3.0.1}$$

where $\Pi_1, \Pi_2 P : H \to H$ and Π_1 is a projection from H to the subspace $\{u \in H : Pu = 0 \text{ } in \text{ } D\}$ of H. Then one can hope that, under reasonable conditions, the element $\Pi_2 f$ defines a solution of the equation $Pu = f$ in D.

For instance, such an approach was successfully tested on the Cauchy- Riemann system $\overline{\partial}$ in $\mathbb{C}^n$ $(n > 1)$ and formulae of the type (3.0.1) were obtained in [AYu], [HeLe] (see also [He]) by the method of integral representations. The construction of formula (3.0.1) by the method of integral representations demands the construction of special holomorphic kernels for the integral Π_1, essentially depending on the domain D.

In this chapter we exploit another idea which was first introduced in complex analysis.

In 1978 two papers of A.V. Romanov devoted to the iterations of the Martinelli -Bochner integral were published (see [Rom1],[Rom2]). In particular, in [Rom2] the following result was obtained.

Theorem (A.V. Romanov [Rom2]). *Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected boundary ∂D of class C^1, and let M be the Martinelli-Bochner integral (on ∂D) defined on the Sobolev space $W^{1,2}(D)$. Then, in the strong operator topology in $W^{1,2}(D)$, $\lim_{\nu \to \infty} M^\nu = \Pi_1$ where Π_1 is a projection from $W^{1,2}(D)$ onto the closed subspace of holomorphic $W^{1,2}(D)$ -functions.*

Using this theorem Romanov (see [Rom2]) obtained a multi-dimensional analogue of the Cauchy-Green formula in the plane (see, for example, [HeLe]), i.e. a formula of the type (3.0.1), and, as a consequence, an explicit formula representing a solution $u \in W^{1,2}(D)$ of the equation $\overline{\partial} u = f$ in the case where D is a pseudo-convex domain with a smooth boundary, and f is a $\overline{\partial}$ -closed (0,1)-form with coefficients in $W^{1,2}(D)$.

Green's integrals (see, §§1.1, 1.2) associated to systems of linear differential equations with injective symbols are natural analogues of the Martinelli-Bochner integral. Within this more general context we obtain in the present chapter the possibility of proving a result similar to the theorem of Romanov and give then some applications.

The plan of the chapter is the following.

The scheme of the proof of the theorem on iterations for Green's integrals is described in §3.1. This scheme is a variation of the original proof by A.V. Romanov [Rom2]. Also some immediate consequences of this theorem are shown in this section.

In §3.2 the theorem on iterations is established for Green's integrals (associated to differential operators with injective symbols) which are constructed by means of special left fundamental solutions (Green's functions).

Using results of §3.2, in §3.3 we obtain solvability conditions for equation $Pu = f$ in the case where the operator P is overdetermined elliptic.

In §3.4 we study the first Sobolev cohomology group of elliptic differential complexes. In particular we obtain criterions for its vanishing.

In §3.5 we obtain necessary and sufficient conditions for the solvability in the Sobolev spaces of a P-Neumann problem for elliptic differential operators.

After discussing in §3.6 some examples of P-Neumann problems, we consider in §3.7 some applications of the Theorem on iterations to the Cauchy and Dirichlet problems.

Sections §3.5 and §3.7 were inspired by results of Kytmanov [Ky] for the multidimensio- nal Cauchy-Riemann system.

Finally in §3.8 we consider the special case of matrix factorizations of the Laplace operator in $\mathbb{R}^n$. æ

§3.1. A theorem on iterations

Let, as above, $X \subset \mathbb{R}^n$ be an open set, $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ be (trivial) vector bundles over X. Let now $P \in do_p(E \to F)$ and let us denote by $\Delta \in do_{2p}(E \to E)$ the differential operator P^*P. As we said before, the operator Δ is a determined elliptic operator of order $2p$ if and only if P is elliptic of order p. We assume that Δ is elliptic and has a bilateral fundamental solution Φ on X. This is always the case if we allow X to be taken sufficiently small or when we assume that the coefficients of P are real analytic. Then $\mathcal{L}(x, y) = {}^tP^*(y, D)\Phi(x, y)$ is a left fundamental solution of $P(x, D)$ on X.

Let D be an (open) relatively compact domain in X, with smooth boundary ∂D as in §1.1. Having fixed a Dirichlet system $\{B_j\}_{j=0}^{p-1}$ of order $(p-1)$ on ∂D as in Definition 1.1.4, we denote by G_P corresponding Green's operator given by Lemma 1.1.6. Then we define the operators M and T by setting, for $u \in W^{p,2}(E_{|D}), f \in L^2(F_{|D})$,

$$(\mathcal{G}u)(x) = -\int_{\partial D} G_P({}^tP^*(y, D)\Phi(x, y), u(y)) \quad (x \in X \backslash \partial D),$$

$$(3.1.1) \qquad (Tf)(x) = \int_D < {}^tP^*(y, D)\Phi(x, y), f(y) >_y dy \quad (x \in X).$$

By Theorem 1.1.7, we have

$$(3.1.2) \qquad (\mathcal{G}u)(x) + (TPu)(x) = \begin{cases} u(x), x \in D, \\ 0, x \in X \backslash \overline{D} \end{cases}$$

for every $u \in W^{p,2}(E_{|D})$

Analogous to the Martinelli- Bochner integral, for every $u \in W^{p,2}(E_{|D})$ the integral $\mathcal{G}u$ defines a $W^{p,2}(E_{|D})$-section which is only "harmonic", i.e. $\Delta \mathcal{G}u = 0$ everywhere outside of ∂D, while in general $P\mathcal{G}u \neq 0$. By Corollary 1.1.9 we have

Proposition 3.1.1. *Let $\partial D \in C^q$ ($q = max(p, 1)$ if $m = p$, and $q = \infty$ if $m > p$). Then the integrals $\mathcal{G}$ and TP given above define linear bounded operators from $W^{m,2}(E_{|D})$ to $W^{m,2}(E_{|D})$ ($m \geq p$).*

In particular, it is possible to consider iterations $\mathcal{G}^\nu = \mathcal{G} \circ \mathcal{G} \circ \cdots \circ \mathcal{G}$, $(TP)^\nu = TP \circ TP \circ \cdots \circ TP$ (ν times) of the integrals $\mathcal{G}$ and TP in the Sobolev spaces $W^{m,2}(E_{|D})$ ($m \geq p$, $\nu \geq 1$).

In order to prove his theorem on iterations for the Martinelli -Bochner integral A.V. Romanov constructed in [Rom2] a suitable scalar product in the space $W^{1,2}(D)$. We follow his approach in our more general case.

Let H be a Hilbert space with a scalar product $(.,.)_H$, and $A : H \to H$, $B : H \to H$ be bounded linear operators with $A + B = Id$ (where Id stands for the identity operator on H). Let us assume that we can construct in the Hilbert space H a scalar product $H(.,.)$ for which the following properties hold:

(3.1.A) *For every $u \in H$: $H(Au, u) \geq 0$, $H(Bu, u) \geq 0$.*

(3.1.B) *The topologies induced in H by $H(.,.)$ and by the initial scalar product $(.,.)_H$ are equivalent.*

In §3.2, by choosing special fundamental solutions, we will construct such a scalar product $H_p^P(.,.)$ for the operators $\mathcal{G}$ and TP in the Hilbert space $W^{p,2}(E_{|D})$ (see also §3.8 for operators $\mathcal{G}$ and TP, associated with matrix factorizations of the Laplace operator in a ball in $\mathbb{R}^n$ and standard fundamental solution of the Laplace operator). In the remaining part of this section we will show that existence of a scalar product with properties (3.1.A) and (3.1.B) implies the convergence of the iterations A^ν and B^ν (cf. [Sh3]).

The kernels $\ker A$ and $\ker B$ of the operators A and B are closed subspaces of H, therefore they are Hilbert spaces (with the Hermitian structure induced from H). If S is a closed subspace of H, we denote by $\Pi(S)$ the orthogonal projection with respect to $H(.,.)$ from H to S.

Theorem 3.1.2. *Assume that a scalar product H, for which (3.1.A) and (3.1.B) hold, is defined in the space H. Then*

$$\lim_{\nu\to\infty} A^\nu = \Pi(\ker B), \quad \lim_{\nu\to\infty} (B)^\nu = \Pi(\ker A)$$

in the strong operator topology in H.

Proof. By (3.1.B) the space H, with the scalar product $H(.,.)$, is a complex Hilbert space. Then (3.1.A) and the fact that $A+B = Id$ imply that the operators A and B are self- adjoint in H with respect to the scalar product $H(.,.)$, and that $0 \leq A \leq Id, 0 \leq B \leq Id$.

The spectral theorem for bounded self -adjoint operators yields

$$A^\nu = \int_0^1 \lambda^\nu dE_\lambda, \ B^\nu = \int_0^1 (1-\lambda)^\nu dE_\lambda \tag{3.1.3}$$

where $\{E_\lambda\}_{0\le\lambda\le 1}$ is a resolution of the identity in the Hilbert space H corresponding, for example, to the operator A and the scalar product $H(.,.)$.

Passing to the limit in (3.1.3) one obtains

$$\lim_{\nu\to\infty} A^\nu = \widetilde{E}_1, \quad \lim_{\nu\to\infty} B^\nu = \widetilde{E}_0$$

where $\widetilde{E}_0 = E_{+0} - E_{-0}, \widetilde{E}_1 = E_{1+0} - E_{1-0}$ are the orthogonal projections from H onto the subspaces $V(0)$,$V(1)$ corresponding to the eigenvalues 0 and 1 of the operator A. Finally, because $A + B = Id$, $V(0) = \ker A, V(1) = \ker B$. □

COROLLARY 3.1.3. *Under the hypotheses of Theorem 3.1.2, for every $u \in H$ the following formulae hold:*

$$u = \lim_{\nu\to\infty} A^\nu u + \sum_{\mu=0}^{\infty} A^\mu B u, \tag{3.1.4}$$

$$u = \lim_{\nu\to\infty} B^\nu u + \sum_{\mu=0}^{\infty} B^\mu A u, \tag{3.1.5}$$

where the limits and the series im the right hand sides converge in the H-norm.

PROOF. Formula $A + B = Id$ implies that for every $\nu \in \mathbb{N}$

$$u = A^\nu u + \sum_{\mu=0}^{\nu-1} A^\mu (Bu) = B^\nu u + \sum_{\mu=0}^{\nu-1} B^\mu (Au). \tag{3.1.6}$$

Using Theorem 3.1.2 we can pass to the limit for $\nu \to \infty$ in (3.1.6), obtaining (3.1.4) and (3.1.5). □

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§3.2. Construction of projection $\Pi(S_P^{p,2}(D))$

In this section we construct a scalar product $H_p^P(.,.)$ on $W^{p,2}(E_{|D})$ satisfying (3.1.A), (3.1.B) for the operators $\mathcal{G}$, TP. This will be obtained by the use of a fundamental solution of $\Delta = P^*P$ enjoying special properties at the boundary of a subdomain Y of X.

Throughout this section we will assume that D is a relatively compact connected open subset of X, with a smooth boundary ∂D of class C^∞. Since Green's integrals do not depend on the choice of the Dirichlet system $\{B_j\}$ on ∂D, in this section we can as well set $B_j = I_k \frac{\partial^j}{\partial n^j}$.

PROPOSITION 3.2.1. *Assume that the operator $\Delta \in do_{2p}(E \to E)$ admits a bilateral fundamental solution Φ on X. Then for every domain $Y \Subset X$, with $\partial Y \in C^\infty$, there exists a unique bilateral fundamental solution $\Phi_Y(x,y)$ of the operator Δ in Y such that*

1. *Φ_Y extends to a smooth function on $(\overline{Y} \times Y)\backslash\{(x,x)|x \in Y\}$;*
2. *$(\frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi_Y(x,y))_{|x\in\partial Y} = 0$ for every $y \in Y$, every multi-index α, and $0 \le j \le p-1$.*

Moreover, the function $\gamma = \Phi - \Phi_Y$ *extends to a smooth function on* $(\overline{Y} \times Y) \cup (Y \times \overline{Y})$.

PROOF. The proof of Proposition 3.2.1 relies on the fact that the existence of a bilateral fundamental solution Φ of Δ in X implies existence and uniqueness of the Dirichlet problem for Δ on every subdomain D of X:

LEMMA 3.2.2. *For every* $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1,\ m \ge p)$ *there exists a (unique) section* $\psi \in S^{m,2}_{\Delta}(D)$ *such that* $(B_j\psi)_{|\partial D} = \psi_j$ $(0 \le j \le p-1)$.

PROOF. Let $\psi \in S^{m,2}_{\Delta}(D)$ be such that $B_j\psi = 0$ on ∂D $(0 \le j \le p-1)$. Then there is a sequence $\{\psi_\nu\}$ of smooth functions with compact support in D such that $\lim_{\nu\to\infty} \psi_\nu = \psi$ in the $W^{p,2}(E_{|D})$-norm. Now using Stokes' formula, one has:

$$0 = \int_D (\psi, \Delta\psi)_x dx = \lim_{\nu\to\infty} \int_D (\psi_\nu, \Delta\psi)_x dx =$$

$$= \lim_{\nu\to\infty} \int_D (P\psi_\nu, P\psi)_x dx = \int_D (P\psi, P\psi)_x dx.$$

Hence $\psi \in S^{m,2}_P(D)$. By Theorem 1.1.7 we obtain that $\psi = M\psi = 0$ in the domain D. This proves the uniqueness of the Dirichlet problem.

We denote by $W^{p,2}_o(E_{|D})$ the space

$$W^{p,2}_o(E_{|D}) = \{u \in W^{p,2}(E_{|D}) : B_j u = 0 \ on\ \partial D\ for\ 0 \le j \le p-1\}.$$

Because Δ is elliptic, we have the classical Gårding inequality:

$$\|u\|^2_{W^{p,2}(E_{|D})} \le c_0 \int_D (Pu, Pu)_x dx + \lambda_0 \|u\|^2_{L^2(E_{|D})} \quad (u \in W^{p,2}_o(E_{|D}))$$

for constants $c_0, \lambda_0 > 0$ which do not depend on u.

As we noted before, Theorem 1.1.7 implies that $u = 0$ if $u \in W^{p,2}_o(E_{|D})$ and $Pu = 0$ in D. Let us prove now that we can find a constant $c > 0$ such that for every $u \in W^{p,2}_o(E_{|D})$ we have

$$\|u\|^2_{W^{p,2}(E_{|D})} \le c \int_D (Pu, Pu)_x dx.$$

We argue by contradiction. If there is no such a constant then we can find a sequence $\{u_\nu\} \subset W^{p,2}_o(E_{|D})$ such that

$$\|u_\nu\|_{W^{p,2}(E_{|D})} = 1, \quad \|Pu_\nu\|_{L^2(F_{|D})} < 2^{-\nu}.$$

Because the unit ball in a separable Hilbert space is weakly compact, we can assume that the sequence $\{u_\nu\}$ weakly converges to a section $u_\infty \in W^{p,2}_o(E_{|D})$. Clearly we have $Pu_\infty = 0$ in D and hence $u_\infty = 0$ by the discussion above. But the Gårding inequality yields

$$1 \le 2^{-\nu} + \lambda_0 \|u_\nu\|_{L^2(E_{|D})}\ for\ every\ \nu$$

and hence, because the inclusion $W^{p,2}_o(E_{|D}) \to L^2(E_{|D})$ is compact, and thus u_ν strongly converges to u_∞ in $L^2(E_{|D})$, we obtain

$$\|u_\infty\|_{L^2(E_{|D})} \ge \lambda_0^{-1}$$

contradicting $u_\infty = 0$.

Thus we proved that the Hermitian form

$$\int_D (Pu, Pv)_x dx$$

defines in the Hilbert space $W_o^{p,2}(E_{|D})$ a scalar product which is equivalent to the original one. Therefore for every $\varphi \in W^{-p,2}(E_{|D})$ there is a unique solution of

$$\begin{cases} u \in W_o^{p,2}(E_{|D}) \\ \int_D (Pu, Pv)_x dx = \varphi(\overline{v}) \ \mathit{for\ every}\ v \in W_o^{p,2}(E_{|D}) \end{cases} \tag{3.2.1}$$

Moreover, by the regularity theorem for elliptic systems, if $\varphi \in W^{m,2}(E_{|D})$, the solution u of (3.2.1) belongs to $W_o^{p,2}(E_{|D}) \cap W^{2p+m,2}(E_{|D})$.

Given $w \in W^{m,2}(E_{|D})$, with $m \geq p$, the map

$$\mathcal{D}(E_{|D}) \ni v \to \int_D (w, \Delta v)_x dx$$

extends to a continuous anti-$\mathbb{C}$-linear functional on $W_o^{p,2}(E_{|D})$ and defines an element $\varphi \in W^{m-2p,2}(E_{|D})$. If u is a solution of (3.2.1) for w, then $\psi = w - u \in W^{m,2}(E_{|D})$, $\Delta\psi = 0$ in D, and $B_j\psi = B_j w$ on ∂D.

The proof of Lemma 3.2.2 is complete. □

Using the lemma, we obtain the fundamental solution Φ_Y in Y by subtracting from Φ the solution γ of the Dirichlet problem

$$\begin{cases} \Delta(x)\gamma(x,y) = 0, \ x \in Y, y \in Y, \\ \frac{\partial^j}{\partial n_x^j}\gamma(x,y) = \frac{\partial^j}{\partial n_x^j}\Phi(x,y), \ x \in \partial Y, y \in Y, \ \ (0 \leq j \leq p-1). \end{cases}$$

The solution smoothly depends on $y \in Y$ and one easily checks that $\Phi_Y = \Phi - \gamma$ satisfies the conditions sets in the statement.

We turn now to the proof of the regularity of γ.

The fact that $\gamma \in C^\infty(\overline{Y} \times Y)$ follows from the regularity up to the boundary of the solution of a Dirichlet problem with smooth data. The regularity of γ in $Y \times \overline{Y}$ is a consequence of the interior regularity of solutions of elliptic systems and the existence and uniqueness results for the Dirichlet problem in Sobolev spaces of negative order (cf. [LiMg], ch. 2, §6).

Let ρ be a defining function for Y. For every nonnegative integer r, define the spaces

$$\Xi^r(E_{|Y}) = \{u \in L^2(E_{|Y}) : \rho^{|\alpha|} D^\alpha u \in L^2(E_{|Y}) \ \mathit{for}\ |\alpha| \leq r\}.$$

They are Hilbert spaces with the norm

$$\|u\|_{\Xi^r(E_{|Y})} = \sum_{|\alpha| \leq r} \|\rho^{|\alpha|} D^\alpha u\|_{L^2(E_{|Y})}.$$

Then $\Xi^{-r}(E_{|Y})$ is defined as the strong dual of $\Xi^r(E_{|Y})$: it can be identified to a subspace of $D'(E_{|Y})$ because $D(E_{|Y})$ is dense in $\Xi^r(E_{|Y})$ for every integer $r \geq 0$. The definition of $\Xi^r(E_{|Y})$ for general $r \in \mathbb{R}$ is obtained by interpolation.

Next we introduce the Hilbert spaces

$$D_\Delta^{-r}(Y) = \{u \in W^{-r,2}(E_{|Y}) : \Delta u \in \Xi^{-r-2p}(E_{|Y})\},$$

endowed with the graph norm, for $r \geq 0$.

By the trace theorem (Theorem 6.5, p. 187 in [LiMg]) the map

$$C^\infty(E_{|\overline{Y}}) \ni u \to \oplus_{j=0}^{p-1}(B_j u) \in \oplus_{j=0}^{p-1}(C^\infty(E_{|\partial Y}))$$

uniquely extends to a continuous linear map

$$D_\Delta^{-r}(Y) \ni u \to \oplus_{j=0}^{p-1}(B_j u) \in \oplus_{j=0}^{p-1}(W^{-r-j-1/2,2}(E_{|\partial Y}))$$

where $r + 1/2 \notin \mathbb{Z}$ and in this case the Dirichlet problem

$$\begin{cases} \Delta u = f \quad in\ Y, \\ \frac{\partial^j}{\partial n^j} u = \psi_j \ on\ \partial Y,\ for\ 0 \leq j \leq p-1, \\ u \in D_\Delta^{-r}(Y) \end{cases}$$

has a unique solution for $f \in \Xi^{-r-2p}(E_{|Y})$ and $\psi_j \in W^{-r-j-1/2,2}(E_{|\partial Y}))$ (this is Theorem 6.6, p. 190 in [LiMg]).

To apply the general result to our special situation, we note that for every fixed $\varepsilon > 0$, and every multi-index α

$$Y \ni x \to D_y^\alpha \Phi(x, y)$$

defines an element of $W^{2p-n/2-|\alpha|-\varepsilon,2}(E_{|Y})$ and $\Delta(x) = D_y^\alpha \delta(x-y) \otimes Id_E$ belongs to $\Xi^{n/2-|\alpha|-\varepsilon}(E_{|Y})$, uniformly for $y \in \overline{Y}$.

Having fixed α, we choose $r_\alpha \geq 0$ with $r_\alpha < 2p - n/2 - |\alpha|$ and $r_\alpha + 1/2 \notin \mathbb{Z}$. Since $\{D_y^\alpha \Phi(x, y) | y \in \overline{Y}\}$ is bounded in $D_\Delta^{-r_\alpha}(Y)$, also $\{\frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi(x, y) | y \in \overline{Y}\}$ is bounded in $\oplus W^{-r_\alpha - j - 1/2,2}(E_{|\partial Y})$.

If $\widetilde{\gamma_\alpha}$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta(x)\widetilde{\gamma_\alpha}(x, y) = 0,\ x \in Y, y \in Y, \\ \frac{\partial^j}{\partial n_x^j}\widetilde{\gamma_\alpha}(x, y) = \frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi(x, y),\ x \in \partial Y, y \in Y, \quad (0 \leq j \leq p-1). \\ \widetilde{\gamma_\alpha}(., y) \in D_\Delta^{-r_\alpha}(Y), \end{cases}$$

then $D_x^\beta \widetilde{\gamma_\alpha}$ is a bounded function of $y \in \overline{Y}$ for every multi-index β while x belongs to a compact subset of Y. Since $\widetilde{\gamma_\alpha} = D_y^\alpha \gamma(x, y)$ for $y \in Y$, the last part of the statement follows. $\square$

Remark 3.2.3. In fact, one could prove more precise regularity of γ outside of diagonal of $\partial Y \times \partial Y$, together with bounds for the growth of its derivatives when (x, y) approaches the singularities (cf. [Shi], p. 145). However, the results obtained above suffices for our purposes.

We fix a domain Y with a C^∞-smooth boundary ∂Y such that $D \Subset Y \Subset X$. Let $\widetilde{S}_\Delta^{m,2}(Y \backslash \overline{D})$ $(m \geq p)$ be the Hilbert space of functions $v \in S_\Delta^{m,2}(Y \backslash \overline{D})$ such that $\frac{\partial^j v}{\partial n^j} = 0$ on ∂Y $(0 \leq j \leq p-1)$. We obtain a linear isomorphism

$$\widetilde{S}_\Delta^{m,2}(Y \backslash \overline{D}) \ni v \xrightarrow{\mathcal{R}^+} \oplus_{j=0}^{p-1}(B_j v)_{\partial D} \in \oplus_{j=0}^{p-1}(W^{m-j-1/2,2}(E_{|\partial D})).$$

Composing $(\mathcal{R}^+)^{-1}$ with the trace operator

$$W^{m,2}(E_{|D}) \ni u \xrightarrow{\mathcal{R}^-} \oplus_{j=0}^{p-1}(B_j u)_{\partial D} \in \oplus_{j=0}^{p-1}(W^{m-j-1/2,2}(E_{|\partial D}))$$

we obtain a continuous linear map

$$W^{m,2}(E_{|D}) \ni u \to S(u) \in \widetilde{S}_{\Delta}^{m,2}(Y\backslash\overline{D}).$$

For $u \in W^{p,2}(E_{|D})$, $f \in L^2(F_{|D})$, and $g \in L^2(F_{|Y\backslash D})$ we introduce now the following notations:

$$\mathcal{G}_Y u(x) = -\sum_{j=0}^{p-1}\int_{\partial D} < (C_j{}^t P^*)(y)\Phi_Y(x,y), B_j u >_y ds \ (x \in Y\backslash\partial D),$$

$$\mathcal{G}_Y S(u)(x) = -\sum_{j=0}^{p-1}\int_{\partial D} < (C_j{}^t P^*)(y)\Phi_Y(x,y), B_j S(u) >_y ds \ (x \in Y\backslash\partial D),$$

$$T_Y f(x) = \int_D < {}^t P^*(y)\Phi_Y(x,y), f(y) >_y dy \ (x \in Y),$$

$$T_Y g(x) = \int_{Y\backslash D} < {}^t P^*(y)\Phi_Y(x,y), g(y) >_y dy \ (x \in Y).$$

Because $(B_j S(u))_{|\partial D} = (B_j u)_{|\partial D}$ $(0 \le j \le p-1)$, we have $\mathcal{G}_Y u = \mathcal{G}_Y S(u)$.

In order to prove the Theorem on the limit of iterations of the integrals $\mathcal{G}_Y$ and $T_Y P$, we consider, for $u, v \in W^{m,2}(E_{|D})(m \ge p)$, the Hermitian form

$$H_p^P(u,v) = \int_D (Pu, Pv)_x dx + \int_{Y\backslash D}(PS(u), PS(v))_x dx. \tag{3.2.2}$$

Proposition 3.2.4. *The Hermitian form $H_p^P(.,.)$ is a scalar product in $W^{m,2}(E_{|D})$.*

Proof. The coefficients of the operator P are $C^\infty(\overline{Y})$ - functions, therefore, $PS(u) \in W^{m-p,2}$ $(E_{|Y\backslash D})$. Then, since $(.,.)_x$ is a Hermitian metric, to prove the statement it is sufficient to prove that $H_p^P(u,u) = 0$ implies $u \equiv 0$ in D.

If $H_p^P(u,u) = 0$ then $u \in S_P^{m,2}(D)$, $S(u) \in S_P^{m,2}(Y\backslash\overline{D})$, and, by definition $(B_j u)_{|\partial D} = (B_j S(u))_{|\partial D}$ $(0 \le j \le p-1)$. Then Theorem 3.2 of [T4] implies that there exists a section $\mathfrak{U} \in S_P(Y)$ such that $\mathfrak{U}_{|D} = u, \mathfrak{U}_{|Y\backslash\overline{D}} = S(u)$. Then $\mathfrak{U} \in S_P^{m,2}(Y)$ and $\frac{\partial^j \mathfrak{U}}{\partial n^j} = 0$ for $0 \le j \le p-1$ on ∂Y. Therefore $\mathfrak{U} \equiv 0$ in Y (by the representation formula proved in Theorem 1.1.7), and in particular $u \equiv 0$ in D. $\square$

Lemma 3.2.5. *Let $(Tf)^- = (Tf)_{|D}$, $(Tf)^+ = (Tf)_{|X\backslash\overline{D}}$. Then for every $f \in W^{p,2}(F_{|D})$ we have*

$$(B_j(Tf)^-)_{|\partial D} - (B_j(Tf)^+)_{|\partial D} = 0,$$

$$({}^t C_j^* P(Tf)^-)_{|\partial D} - ({}^t C_j^* P(Tf)^+)_{|\partial D} = ({}^t C_j^* f)_{|\partial D}.$$

PROOF. Using Stokes' formula we obtain for $x \notin \partial D$ and $f \in W^{p,2}(E_{|D})$:
(3.2.3)

$$Tf(x) = \int_D < \Phi(x,y), P^* f(y) >_y dy - \int_{\partial D} \sum_{j=0}^{p-1} < {}^t B_j^*(y)\Phi(x,y), {}^t C_j{}^*(y) f(y)).$$

Because $P^* f \in L^2(E_{|D})$, the first integral in the right hand side defines a section in $W^{2p,2}(E_{|Y})$. Indeed the fundamental solution Φ is a pseudo-differential operator of order $(-2p)$ on X. Thus it does not contribute to the jumps of the derivatives of Tf on ∂D up to order $(2p-1)$. The statement of the lemma is then a consequence of the jump formula (1.3.5), after nothing that $\{-C_j{}^t P^*, {}^t B_j^*\}_{j=0}^{p-1}$ is the Dirichlet system corresponding to the Dirichlet system $\{B_j, {}^t C_j^* P\}_{j=0}^{p-1}$ with respect to Δ in Lemma 1.1.6 (see Theorem 1.4.4). □

Remark 3.2.6. In particular, if $f \in W^{p,2}(F_{|D})$ has compact support in D, then $Tf \in W^{2p,2}(E_{|Y})$.

Let $(T_Y g)^+ = (T_Y g)_{|Y \setminus \overline{D}}$, $(T_Y g)^- = (T_Y g)_{|D}$, and introduce similar notations for $T_Y f$ $(f \in L^2(F_{|D}),\ g \in L^2(F_{|Y\setminus D}))$.

LEMMA 3.2.7. *Let $r \geq 0$, $\partial D \in C^q$ ($q = 1$ if $r = 0$, $q = \infty$ if $r > 0$). Then there exist a positive number $c(r)$ such that for every $f \in W^{r,2}(F_{|D})$ and $g \in W^{r,2}(F_{|Y\setminus D})$*

$$\|(T_Y f)^-\|^2_{W^{p+r,2}(E_{|D})} \leq c(r) \|f\|^2_{W^{r,2}(F_{|D})},$$

$$\|(T_Y f)^+\|^2_{W^{p+r,2}(E_{|Y\setminus D})} \leq c(r) \|f\|^2_{W^{r,2}(F_{|D})},$$

$$\|(T_Y g)^-\|^2_{W^{p+r,2}(E_{|D})} \leq c(r) \|g\|^2_{W^{r,2}(F_{|Y\setminus D})}.$$

PROOF. By Proposition 3.2.1, $\gamma = \Phi - \Phi_Y$ is smooth in $(\overline{Y} \times Y) \cup (Y \times \overline{Y})$. Then

$$L^2(F_{|D}) \ni f \to \int_D < {}^t P^*(y)\gamma(x,y), f(y) >_y dy \in C^\infty(E_{|\overline{Y}})$$

and

$$L^2(F_{|Y\setminus\overline{D}}) \ni g \to \int_{Y\setminus D} < {}^t P^*(y)\gamma(x,y), g(y) >_y dy \in C^\infty(E_{|\overline{D}})$$

are linear and continuous maps. Therefore the proof of the estimates is reduced to the proof of the analogous estimates for T substituting T_Y.

When $0 \leq r < 1/2$, the estimates hold true because ${}^t P^* \Phi(x,y)$ is a pseudo-differential operator of order $(-p)$ on X and for general $r > 0$ by nothing that it has moreover the transmission property relative to every relatively compact open subset of X with a smooth boundary (cf. [ReSz], 2.2.2 and 2.3.2.4). □

Remark 3.2.8. The lemma, together with the preceeding remark, implies that $T_Y f \in W^{p,2}(E_{|Y})$ for every $f \in L^2(F_{|D})$. Indeed we can approximate $f \in L^2(F_{|D})$ by smooth sections with compact support in D in the L^2-norm. By the jump Lemma 3.2.5, $(T_Y f)^-$ and $(T_Y f)^+$ agree with their derivatives up to order $(p-1)$ on ∂D when f is smooth with compact support in D and hence by continuity the same is true when $f \in L^2(F_{|D})$.

PROPOSITION 3.2.9. *For every* $u, v \in W^{p,2}(E_{|D})$, $f \in L^2(F_{|D})$

$$H_p^P(T_Y f, v) = \int_D (f, Pv)_x dx,$$

$$H_p^P(\mathcal{G}_Y u, v) = \int_{Y \setminus D} (PS(u), PS(v))_x dx.$$

PROOF. By integration by parts we obtain (cf. Lemma 1.1.6)

$$\int_D (f, Pv)_x dx - \int_D (P^* f, v)_x dx =$$

(3.2.4)
$$= \sum_{j=0}^{p-1} \int_{\partial D} < *_{F_j} B_j v, {}^t C_j^* f >_x \, ds \; for \; every \; f \in W^{p,2}(F_{|D}), \, v \in W^{p,2}(E_{|D})$$

and analogously

$$\int_{Y \setminus D} (PS(u), PS(v))_x dx = -\sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) S(v), {}^t C_j^* PS(u) >_y \, ds = \tag{3.2.5}$$

$$= -\sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) v, {}^t C_j^* PS(u) >_y \, ds \; for \; every \; u, v \in W^{p,2}(E_{|D}).$$

Let $u \in W^{2p,2}(E_{|D})$, $v \in W^{p,2}(E_{|D})$, and apply formula (3.2.4) for $f = Pu$. Then we obtain, using (3.2.4) and (3.2.5)

$$H_p^P(u, v) = \sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) v, {}^t C_j^* Pu - {}^t C_j^* PS(u) >_y \, ds + \int_D (P^* Pu, v)_y dy.$$

Let $f \in \mathcal{D}(F_{|D})$. Then we can substitute $T_Y f$ for u in the formula above, to obtain

$$H_p^P(T_Y f, v) = \int_D (P^* P T_Y f, v)_y dy +$$

$$+ \sum_{j=0}^{p-1} \int_{\partial D} < *_{F_j} B_j v, {}^t C_j^* P(T_Y f)^- - {}^t C_j{}^* P(T_Y f)^+ >_y \, ds.$$

By Remark 3.2.6, $T_Y f \in W^{2p,2}(E_{|D})$ and thus the second summand in the right hand side of the last equality equals to zero. Because

$$P^* P T_Y f(x) = P^* f(x) \; (x \in D),$$

we get

$$H_p^P(T_Y f, v) = \int_D (P^* f, v)_y dy = \int_D (f, Pv)_y dy.$$

Since $\mathcal{D}(F_{|D})$ is dense in $L^2(F_{|D})$, this formula holds for every $v \in W^{p,2}(E_{|D})$ and every $f \in L^2(F_{|D})$.

Finally, (3.1.2) implies that

$$H_p^P(\mathcal{G}_Y u, v) = H_p^P(u - T_Y P u, v) = \int_{Y \setminus D} (PS(u), PS(v))_y dy. \qquad \square$$

LEMMA 3.2.10. *For every* $u \in W^{m,2}(E_{|D})$ $(m \geq p)$

$$(T_Y Pu)(x) + (T_Y PS(u))(x) = \begin{cases} u(x), x \in D, \\ S(u)(x), x \in Y \backslash \overline{D}. \end{cases}$$

PROOF. Since $\overline{Y} \subset X$, Theorem 1.1.7 implies that

$$- \int_{\partial(Y\backslash D)} G_P({}^tP^*(y)\Phi(x,y), S(u)(y)) + \int_{Y\backslash D} < {}^tP^*(y)\Phi(x,y), PS(u)(y) >_y dy =$$

$$= \begin{cases} S(u)(x), x \in Y\backslash D, \\ 0, x \in X\backslash(\overline{Y\backslash D}). \end{cases}$$

On the other hand, if $\gamma = \Phi - \Phi_Y$ then for every fixed point $x \in Y$ the integrals

$$\int_{\partial(Y\backslash D)} G_P({}^tP^*(y)\gamma(x,y), S(u)(y)) \ \textit{and} \ \int_{Y\backslash D} < {}^tP^*(y)\gamma(x,y), PS(u)(y) >_y dy$$

are well defined. Then, since ${}^t\Delta(y)\gamma(x,y) = 0$ for $(x,y) \in Y \times Y$, Stokes' formula yields for $x \in Y$

$$- \int_{\partial(Y\backslash D)} G_P({}^tP^*(y)\gamma(x,y), S(u)(y)) + \int_{Y\backslash D} < {}^tP^*(y)\gamma(x,y), PS(u)(y) >_y dy = 0$$

Therefore, since $\frac{\partial^j S(u)}{\partial n^j} = 0$ on ∂Y

$$(T_Y PS(u))(x) - (\mathcal{G}_Y S(u))(x) = \begin{cases} 0, x \in D, \\ S(u)(x), x \in Y\backslash\overline{D}. \end{cases} \tag{3.2.6}$$

Finally, $(B_j u)_{|\partial D} = (B_j S(u))_{|\partial D}$ by definition, hence $\mathcal{G}_Y u = \mathcal{G}_Y S(u)$. Now adding (3.1.2) and (3.2.6) we obtain the statement. □

LEMMA 3.2.11. *The Hilbert spaces* $S^{m,2}_\Delta(D)$, $\widetilde{S}^{m,2}_\Delta(Y\backslash\overline{D})$, $\oplus_{j=0}^{p-1} W^{m-j-1/2,2}(F_{j\,|\partial D})$ *are topologically isomorphic.*

PROOF. Lemma 3.2.2 implies that for every $\oplus u_j \in \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$ there exist (unique) solutions $u \in S^{m,2}_\Delta(D)$ and $S(u) \in \widetilde{S}^{m,2}_\Delta(Y\backslash\overline{D})$ of the interior and exterior Dirichlet problems. Therefore, in order to prove the statement of the lemma it is sufficient to prove existence of constants $c_i > 0$ $(1 \leq i \leq 4)$ such that for every $\oplus u_j \in \oplus W^{m-j-1/2,2}(F_{j\,\partial D})$

$$c_1 \|u\|^2_{W^{m,2}(E_{|D})} \leq \sum_{j=0}^{p-1} \|u_j\|^2_{W^{m-j-1/2,2}(F_{j\,|\partial D})} \leq c_2 \|u\|^2_{W^{m,2}(E_{|D})}$$

$$c_3 \|S(u)\|^2_{W^{m,2}(E_{|Y\backslash D})} \leq \sum_{j=0}^{p-1} \|u_j\|^2_{W^{m-j-1/2,2}(F_{j\,|\partial D})} \leq c_4 \|S(u)\|^2_{W^{m,2}(E_{|Y\backslash D})}. \tag{3.2.7}$$

The existence of the constants c_2, c_4 follows from the continuity of the restriction maps

$$\mathcal{R}^- : S^{m,2}_\Delta(D) \to \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$$

$$\mathcal{R}^+ : \widetilde{S}^{m,2}_\Delta(Y\backslash\overline{D}) \to \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$$

where $\mathcal{R}^- u = \oplus (B_j u)_{|\partial D}, \mathcal{R}^+ S(u) = \oplus (B_j S(u))_{|\partial D}$ (see [EgSb], p.120). Since $\mathcal{R}^-, \mathcal{R}^+$ are one-to-one (see Lemma 3.2.2), the existence of constants c_1, c_3 follows from the open mapping theorem. □

PROPOSITION 3.2.12. *The topologies induced in* $W^{p,2}(E_{|D})$ *by* $H_p^P(.,.)$ *and by the standard scalar product are equivalent.*

PROOF. Since the coefficients of P are $C^\infty(\overline{Y})$- functions then there are constants $c_5, c_6 > 0$ such that for every $u \in W^{p,2}(E_{|D})$

$$(Pu, Pu)_x \leq c_5 \sum_{|\alpha|\leq p} (D^\alpha u, D^\alpha u)_x, \ (PS(u), PS(u))_x \leq c_6 \sum_{|\alpha|\leq p} (D^\alpha S(u), D^\alpha S(u))_x.$$

On the other hand, Lemma 3.2.11 (see (3.2.7)) implies that for every $u \in W^{p,2}(E_{|D})$, we have $\|S(u)\|^2_{W^{p,2}(E_{Y\setminus\overline{D}})} \leq (c_3)^{-1} \sum_{j=0}^{p-1} \|u_j\|^2_{W^{m-j-1/2,2}(F_{j\,|\partial D})}$. Then, since the restriction mapping $\mathcal{R}^-$ (see proof of Lemma 3.2.11) is continuous, we have $\|S(u)\|^2_{W^{p,2}(E_{Y\setminus\overline{D}})} \leq$ $c_2(c_3)^{-1}\|u\|^2_{W^{p,2}(E_{|D})}$. Hence

$$H_p^P(u,u) \leq (c_5 + c_6 c_2 (c_3)^{-1})\|u\|^2_{W^{p,2}(E_{|D})}.$$

Conversely, Lemmata 3.2.7 and 3.2.10 imply that

$$(1/2)\|u\|^2_{W^{p,2}(E_{|D})} \leq \|T_Y Pu\|^2_{W^{p,2}(E_{|D})} + \|T_Y PS(u)\|^2_{W^{p,2}(E_{|D})} \leq$$

$$\leq c(0)\|Pu\|^2_{L^2(F_{|D})} + c(0)\|PS(u)\|^2_{L^2(F_{|Y\setminus D})} = c(0)H_p^P(u,u),$$

which was to be proved. □

In the following theorem $\widetilde{S}_P^{p,2}(Y\setminus\overline{D})$ stands for the subspace of $W^{p,2}(E_{|D})$ which consists of functions $u \in W^{p,2}(E_{|D})$ such that $PS(u) = 0$ in $(Y\setminus\overline{D})$.

THEOREM 3.2.13. *In the strong operator topology in* $W^{p,2}(E_{|D})$

$$\lim_{\nu\to\infty} \mathcal{G}_Y^\nu = \Pi(S_P^{p,2}(D)),$$

$$\lim_{\nu\to\infty} (T_Y P)^\nu = \Pi(\widetilde{S}_P^{p,2}(Y\setminus\overline{D})).$$

PROOF. First, Propositions 3.2.9 and 3.2.12 imply that (3.1.A) and (3.1.B) hold for the Hermitian $H_p^P(.,.)$ defined in (3.2.2) and the operators $\mathcal{G}_Y$, $T_Y P$. Second, Proposition 3.2.9 implies that $ker T_Y P = S_P^{p,2}(D)$. Third, Proposition 3.2.9, (3.1.2) and Lemma 3.2.10 imply that $\mathcal{G}_Y u = 0$ if and only if $S(u) \in S_P^{p,2}(Y\setminus\overline{D})$. Hence the theorem follows from Theorem 3.1.2. □

Remark 3.2.14. Let the operator P satisfy the so-called Uniqueness Condition in the small on X, i.e. $Pu = 0$ in a domain $D \subset X$ and $u = 0$ in an open subset of D imply $u \equiv 0$ in D. Then, if ∂D is connected, the Uniqueness Theorem for the Cauchy problem for systems with injective symbols (see [ShT2], Theorem 2.8), implies that $\widetilde{S}_P^{p,2}(Y\setminus\overline{D}) = W_0^{p,2}(E_{|D})$. For instance, the Uniqueness Condition holds if the coefficients of the operator P are real analytic.

Remark 3.2.15. Lemmata 3.2.7, 3.2.9 and 3.2.12. implies that the operator $T_Y : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is the adjont operator (in the sence of Hilbert Spaces Theory) to the operator $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ with respect to the scalar product $H_p^P(.,.)$ in $W^{p,2}(E_{|D})$ and the standard scalar product in $L^2(F_{|D})$.

Though we can not constract such a scalar product in $W^{m,2}(E_{|D})$ in general, we will do it for $S_{\Delta_n}^{m,2}(E_{|B_R})$ and a matrix factorization P of the Laplace operator Δ_n in $\mathbb{R}^n$ in §3.8. æ

§3.3. Solvability conditions for the equation $Pu = f$

In this section we will use Theorem 3.2.13 to investigate solvability of equation $Pu = f$. In particular, when $Pu = f$ is solvable we will obtain an expression of the solution by means of a series that can be computed from the data.

Let $P \in do_p(E \to F)$ be an elliptic operator of order p, as in §§3.1, 3.2. We formulate now

PROBLEM 3.3.1. *Let $r \geq 0$, $0 \leq m \leq p + r$, and $f \in W^{r,2}(F_{|D})$ be a given section. It is required to find a section $u \in W^{m,2}(E_{|D})$ such that $Pu = f$ in D.*

We denote by R_Y the series

$$R_Y = \sum_{\mu=0}^{\infty} \mathcal{G}_Y^{\mu} T_Y.$$

For every $r \geq 0$ we set

$$\begin{aligned} domR_Y^{p,r} =& \{g \in L^2(F_{|D}) : \ R_Y g \textit{ converges in the } W^{p,2}(E_{|D}) - \textit{norm}, \\ & \textit{and } P(R_Y g) \in W^{r,2}(F_{|D})\}. \end{aligned}$$

Then R_Y defines a linear operator $R_Y^{p,r} : domR_Y^{p,r} \to W^{p,2}(E_{|D})$. This series will play an essential role in our investigation of equation $Pu = f$.

PROPOSITION 3.3.2. *Let $(S_P^{p,2}(D))^{\perp}$ be the orthogonal complement of the subspace $S_P^{p,2}(D)$ in $W^{p,2}(E_{|D})$ with respect to $H_p^P(\cdot,\cdot)$. Then $Im(R_Y^{(p,0)}) = (S_P^{p,2}(D))^{\perp}$.* $\blacksquare$

PROOF. If $f \in domR_Y^{(p,0)}$ then $R_Y f \in W^{p,2}(E_{|D})$, and, since $\mathcal{G}_Y$ is continuous (see Proposition 3.1.1),

$$\mathcal{G}_Y R_Y f = \mathcal{G}_Y \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu} \mathcal{G}_Y^{\mu} T_Y f = \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu} \mathcal{G}_Y^{\mu+1} T_Y f = R_Y f - T_Y f. \tag{3.3.1}$$

Therefore

$$\mathcal{G}_Y^{\nu} R_Y f = R_Y f - \sum_{\mu=0}^{\nu-1} \mathcal{G}_Y^{\mu} T_Y f. \tag{3.3.2}$$

Passing to the limit for $\nu \to \infty$ in (3.3.2) we obtain that $\lim_{\nu\to\infty} \mathcal{G}_Y^{\nu} R_Y f = R_Y f - R_Y f = 0$, i.e. $\Pi(S_P^{p,2}(D))R_Y f = 0$ and therefore $R_Y f \in (S_P^{p,2}(D))^{\perp}$.

Conversely, if $u \in (S_P^{p,2}(D))^{\perp}$ then (3.1.4) and Theorem 3.2.13 imply that $u = R_Y P u$. By Proposition 3.1.1 and Corollary 3.1.3 we have $Pu \in domR_Y^{(p,0)}$. Therefore $(S_P^{p,2}(D))^{\perp} \subset Im(R_Y^{(p,0)})$. $\square$

In particular Proposition 3.3.2 implies that $Im(R_Y^{(p,r)}) \subset (S_P^{p,2}(D))^{\perp}$.

By formula (3.1.4) the series R_Y defines the left inverse of P on $(S_P^{p,2}(D))^{\perp}$. In the following proposition we find a condition for R_Y to be also a right inverse operator of P.

PROPOSITION 3.3.3. $\ker R_Y^{(p,r)} = 0$ *if and only if* $PR_Y^{(p,r)} = Id_{|domR_Y^{(p,r)}}$.

PROOF. If $f \in domR_Y^{(p,r)}$ then $R_Y f \in W^{p,2}(E_{|D})$ and $PR_Y f \in domR_Y^{(p,r)}$ by (3.1.4). Because $R_Y^{(p,r)}$ is a left inverse of P on $(S_P^{p,2}(D))^{\perp}$ and, due to Proposition 3.3.2, $ImR_Y^{(p,r)} \subset (S_P^{p,2}(D))^{\perp}$, we obtain $R_Y^{(p,r)} PR_Y^{(p,r)} = R_Y^{(p,r)}$. From this identity we deduce that $PR_Y^{(p,r)} = Id_{|domR_Y^{(p,r)}}$ if $R_Y^{(p,r)}$ is injective, while the converse statement is obvious. □

PROPOSITION 3.3.4. $\ker R_Y^{(p,r)} = \ker T_Y \cap domR_Y^{(p,r)}$ $(r \geq 0)$.

PROOF. Clearly $\ker T_Y \subset \ker R_Y^{(p,r)}$. The opposite inclusion follows from (3.3.1). □

The following theorem is rather trivial because we have proved in §3.2 that the operator $T_Y : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is the adjoint operator (in the sense of Hilbert Spaces Theory) to the operator $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ with respect to the scalar product $H_p^P(.,.)$ in $W^{p,2}(E_{|D})$ and the standard scalar product in $L^2(F_{|D})$ (see Remark 3.2.15).

THEOREM 3.3.5. *Let* $r \geq 0$, $m = p$ *and* $f \in W^{r,2}(F_{|D})$. *Then Problem 3.3.1 is solvable if and only if*

(1) $f \subset domR_Y^{p,r}$;
(2) $\int_D (g, f)_x dx = 0$ *for every* $g \in \ker T_Y \cap W^{r,2}(F_{|D})$.

PROOF. Necessity. Let Problem 3.3.1 be solvable. Then Proposition 3.1.3 and Theorem 3.2.13 imply that $Pu = PR_Y Pu$ for $u \in W^{p,2}(E_{|D})$, i.e, (1) holds. On the other hand, due to Proposition 3.2.9, for every $g \in L^2(F_{|D})$ we have:

$$\int_D (g, f)_x dx = \int_D (g, Pu)_x dx = H_p^P(T_Y g, u),$$

i.e. (2) holds.

Sufficiency. Since, under the hypothesis of the theorem, $R_Y f \in W^{p,2}(F_{|D})$, by Proposition 3.3.2

$$R_Y f = \lim_{\nu \to \infty} M_Y^{\nu} R_Y f + R_Y PR_Y f = R_Y PR_Y f. \tag{3.3.3}$$

In particular, $(f - PR_Y f) \in \ker R_Y^{p,r} \cap W^{r,2}(F_{|D})$, and, due to Proposition 3.3.4, $(f - PR_Y f) \in \ker T_Y \cap W^{s,q}(E_{|D})^{r,2}(F_{|D})$. On the other hand, using the hypothesis of the theorem, we conclude that

$$\int_D (f - PR_Y f, f - PR_Y f)_x dx = 0.$$

Therefore $f = PR_Y f$, i.e. Problem 3.3.1 is solvable. □

Let us assume now that P is included into some elliptic complex of differential operators on X:

$$C^{\infty}(E) \xrightarrow{P} C^{\infty}(F) \xrightarrow{P^1} C^{\infty}(G) \tag{3.3.4}$$

for a trivial vector bundle $G = X \times \mathbb{C}^t$ and $P^1 \in do_{p_1}(F \to G)$. The assumptions mean that

$$P^1 \circ P = 0$$

and that

$$E_x \xrightarrow{\sigma_p(P)(x,\zeta)} F_x \xrightarrow{\sigma_{p_1}(P^1)(x,\zeta)} G_x$$

is an exact sequence for every $x \in X$ and $\zeta \in \mathbb{R}^n \backslash \{0\}$. According to [Sa] (cf. also [AnNa]) this is possible under rather general assumptions on P.

Then the condition $P^1 f = 0$ is necessary in order that Problem 3.3.1 be solvable.

Let, as before, $\{B_j\}_{j=0}^{p-1}$ be a Dirichlet system of order $(p-1)$ on ∂D, $\{C_j\}_{j=0}^{p-1}$ be the Dirichlet system associated to $\{B_j\}_{j=0}^{p-1}$ as in Lemma 1.1.6, and let, for $r \geq 0$,

$$\mathfrak{H}^{r,2}(D) = \{g \in W^{r,2}(F_{|D}) \textit{ such that } P^*g = 0,\ P^1 g = 0 \textit{ in } D, \textit{ weakly satisfying}$$

$$\textit{the boundary conditions } ({}^tC_j^* g)_{|\partial D} = 0,\ 0 \leq j \leq p-1\}.$$

We call the $\mathfrak{H}^{r,2}(D)$ *harmonic spaces* (for complex (3.3.4)). By the ellipticity assumptions, $\mathfrak{H}^{r,2}(D) \subset C^\infty(F_{|D})$. It is not difficult to show that for the Dolbeault complex this definition of the harmonic space $\mathfrak{H}^{0,2}(D)$ is equivalent to the one given in [Kohn] (see also [Hö1]).

Let us denote by $\mathfrak{N}_m^{r,2}(D)$ the set of all $f \in W^{r,2}(F_{|D})$ for which Problem 3.3.1 is solvable:

$$\begin{aligned}\mathfrak{N}_m^{r,2}(D) = &\{f \in W^{r,2}(F_{|D}) : \textit{ there exists a section } u \in W^{m,2}(E_{|D}) \\ &\textit{such that } Pu = f \textit{ in } D\}.\end{aligned}$$

We obtain:

Proposition 3.3.6. *We have*

(1) $\mathfrak{N}_m^{r,2}(D) \subset S_{P^1}^{r,2}(D)$ $(m \geq 0)$;
(2) $\int_D (g, f)_x dx = 0$ *for every* $f \in \mathfrak{N}_m^{r,2}(D)$ *and every* $g \in \mathfrak{H}^{r,2}(D)$ $(m \geq p)$;
(3) $\mathfrak{N}_m^{r,2}(D) \subset dom R_Y^{p,r}$ $(m \geq p)$;
(4) $\ker T_Y \cap \mathfrak{N}_m^{r,2}(D) = 0$ $(m \geq p)$.

Proof. (1) is trivial, because (3.3.4) is a complex; and we have proved that (3) holds in Theorem 3.3.5. To prove (2), we fix $f \in \mathfrak{N}_m^{r,2}(D)$ and a section $u \in W^{m,2}(E_{|D})$ such that $Pu = f$ in D.

For $\varepsilon > 0$ we set $D_\varepsilon = \{x \in D : \ dist(x, \partial D) > \varepsilon\}$. Since the differential complex (3.3.4) is elliptic, $\mathfrak{H}^{r,2}(D) \subset C^\infty(F_{|D})$. Hence, for every $g \in \mathfrak{H}^{r,2}(D)$, we have:

$$\int_D (g, f)_x dx = \int_D (g, Pu)_x dx = \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (g, Pu)_x dx =$$

$$= \lim_{\varepsilon \to 0} \left(\int_{D_\varepsilon} (P^* g, u)_x dx - \int_{\partial D_\varepsilon} G_{P^*}(*_E u, g)_x dx \right) =$$

$$= \lim_{\varepsilon \to 0} \sum_{j=0}^{p-1} \int_{\partial D_\varepsilon} < (*_{F_j} B_j u), {}^tC_j^* g >_y ds = 0.$$

Therefore (2) holds.

Finally, if $f \in \ker T_Y \cap \mathfrak{N}_m^{r,2}(D)$ then (due to Proposition 3.3.4) $f \in \ker R_Y^{p,r} \cap \mathfrak{N}_m^{r,2}(D)$. Therefore $0 = PR_Y f = f$. $\square$

THEOREM 3.3.7. *Let $r \geq 0$, $m = p$ and $f \in W^{r,2}(F_{|D})$. Then Problem 3.3.1 is solvable if and only if*

(1) $f \in S^{r,2}_{P^1}(D) \cap dom R^{p,r}_Y$;
(2) $\int_D (g, f)_x dx = 0$ *for every* $g \in \mathfrak{H}^{r,2}(D)$.

PROOF. The necessity follows from Proposition 3.3.6. In order to prove the converse statement we will use the following lemma.

LEMMA 3.3.8. $\mathfrak{H}^{r,2}(D) = \ker T_Y \cap S^{r,2}_{P^1}(D)$ $(r \geq 0)$.

PROOF. Let $f \in \mathfrak{H}^{r,2}(D)$. Then $f \in C^\infty(F_{|D})$. But for every $f \in ker P^* \cap C^\infty(F_{|D}) \cap L^2(F_{|D})$ and $x \in Y \backslash \partial D$ we have:

$$T_Y f(x) = \int_D < {}^tP^*(y)\Phi_Y(x,y), f(y) >_y dy = \lim_{\varepsilon \to 0} \int_{D_\varepsilon} < {}^tP^*(y)\Phi_Y(x,y), f(y) >_y dy =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \sum_{j=0}^{p-1} < {}^tB^*_j(y)\Phi_Y(x,y), {}^tC^*_j f(y) >_y ds. \tag{3.3.5}$$

Therefore, since the weak boundary values $({}^tC^*_j f)_{|\partial D}$ equal to zero $(0 \leq j \leq p-1)$, the last limit in (3.3.5) is equal to zero.

Let us prove now the opposite inclusion. Since Φ_Y is a bilateral fundamental solution of the operator P^*P in Y then $\widetilde{\Phi}_Y(x,y) = \Phi_Y(y,x)$ is a bilateral fundamental solution of the operator ${}^t(P^*P)$ on Y. In particular, for every $v \in \mathcal{D}(E^*_{|D})$ we have

$$v(y) = \int_D < \Phi_Y(x,y), {}^t(P^*P)(x)v(x) >_x dx.$$

For every given section $f \in L^2(F_{|D})$ we can find a sequence $\{f_N\} \subset C(F_{|\overline{D}})$ such that $\lim_{N\to\infty} f_N = f$ in the $L^2(F_{|D})$- norm. Assuime moreover that $f \in \ker T_Y \cap W^{r,2}(F_{|D})$. Then, for every $v \in \mathcal{D}(E^*_{|D})$ we have

$$\int_D < {}^tP^*(y)v(y), f(y) >_y dy = \lim_{N\to\infty} \int_D < {}^tP^*(y)v(y), f_N(y) >_y dy =$$

$$= \lim_{N\to\infty} \int_{D_y} < {}^tP^*(y) \int_{D_x} < \Phi_Y(x,y), {}^t(P^*P)(x)v(x) >_x dx, f_N(y) >_y dy =$$

$$= \lim_{N\to\infty} \int_D < T_Y f_N(x), {}^t(P^*P)(x)v(x) >_x dx.$$

By Lemma 3.2.7, $T_Y : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is continuous and therefore $\{Tf_N\}$ converges in the $W^{p,2}(E_{|D})$-norm to $T_Y f = 0$. This shows that

$$\int_D < {}^tP^*(y)v(y), f(y) >_y dy = 0 \ for \ every \ v \in \mathcal{D}(F_{|D}).$$

Hence $P^*f = P^1 f = 0$ if $f \in \ker T_Y \cap S^{r,2}_{P^1}(D)$. Note that regularity theorem for elliptic systems gives in particular $\ker T_Y \cap S^{r,2}_{P^1}(D) \subset C^\infty(F_{|D})$.

To complete the proof, we only need to show that (in the weak sense) $({}^tC_j^* f)_{|\partial D} = 0$ on ∂D $(0 \leq j \leq p-1)$ for $f \in \ker T_Y \cap S^{r,2}_{P1}(D)$. To this aim, we prove that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^tC_j^* f(y) >_y ds = 0$$

for every $v^{(j)} \in C^\infty_{comp}(F_j^*)$.

Let $v^{(j)} \in C^\infty_{comp}(F_j^*)$. Then we fix a domain Ω with $D \Subset \Omega \Subset Y$, and find a section $v \in \mathcal{D}(E^*_{|\Omega})$ such that ${}^tB_j^* v = v^{(j)}$ on ∂D, and ${}^tB_i^* v = 0$ on ∂D, if $i \neq j$ (see Lemma 1.1.5). Again we use representation formula:

$$v(y) = \int_\Omega < \Phi_Y(x,y), {}^t(P^*P)(x) v(x) >_x dx.$$

Since $P^* f = 0$ and $f \in C^\infty(F_{|D})$, arguing as before we have

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^tC_j^* f(y) >_y ds = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \sum_{i=0}^{p-1} < {}^tB_i^* v, {}^tC_i^* f(y) >_y ds =$$

$$= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} < {}^tP^* v, f >_y dy = \int_D < {}^tP^* v, f >_y dy =$$

$$= \lim_{N \to \infty} \int_\Omega < T_Y f_N(x), {}^t(P^*P)(x) v(x) >_x dx.$$

Lemma 3.2.7 implies that $\lim_{N\to\infty} (T_Y f_N)_{|\Omega}$ converges in $W^{p,2}(F_{|\Omega})$ to $(T_Y f)_{|\Omega}$. Due to Proposition 3.2.1 and Remark 3.2.8, $T_Y f = 0$ in D implies $T_Y f = 0$ in Y. Therefore

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^tC_j^* f(y) >_y ds = 0.$$

The proof of the lemma is complete. □

Now we turn to the proof of Theorem 3.3.7. Using foormula (3.3.3), the hypothesis of the theorem we see (as in the proof of Theorem 3.3.5) that $(f - PR_Y f) \in \ker R_Y^{p,r} \cap W^{r,2}(F_{|D})$. Now, due to Proposition 3.3.4, $(f - PR_Y f) \in \ker T_Y \cap S^{r,2}_{P1}(D)$. On the other hand, using Lemma 3.3.8 and the hypothesis of the theorem, we conclude that

$$\int_D (f - PR_Y f, f - PR_Y f)_x dx = 0.$$

Therefore $f = PR_Y f$, i.e. Problem 3.3.1 is solvable. □

As one can see from the proof of Theorems 3.3.5 and 3.3.7, if the equation $Pu = f$ is solvable in $W^{p,2}(E_{|D})$ then we obtain a formula for a solution of the equation:

$$u = R_Y f = \sum_{\nu=0}^{\infty} \mathcal{G}_Y^\nu T_Y f.$$

In the case where $P = \overline{\partial}$ and $\mathcal{G}_Y$ is the Martinelli- Bochner integral such a formula was obtained by Romanov [Rom2].

We conjecture that when the Poincarè lemma (local solvability) is valid for an elliptic complex, a solution in $W^{p,2}(E_{|D})$ can be found for every datum f in $W^{p,2}(F_{|D})$ satisfying the integrability conditions. If this is the case, the formula above produces rather explicitely a way to obtain a solution by successive approximations.

Remark 3.3.9. Proposition 3.3.2 and Theorem 3.2.13 imply that the solution $u = R_Y f$ of Problem 3.3.1 belongs to $(S_P^{p,2}(D))^\perp$ where $(S_P^{p,2}(D))^\perp$ is the orthogonal (with respect to $H_p^P(.,.)$) complement of $S_P^{p,2}(D)$ in $W^{p,2}(E_{|D})$, and is the unique solution belonging to this subspace.

We note that the general term $\mathcal{G}_Y^\mu T_Y f$ of the series $R_Y f$ is infinitesimal in $W^{p,2}(E_{|D})$ for every $f \in L^2(F_{|D})$. This is a consequence of the theorem on iterations.

PROPOSITION 3.3.10. *For every* $f \in L^2(F_{|D})$, $\lim_{\nu\to 0} \mathcal{G}_Y^\nu T_Y f = 0$ *in the* $W^{p,2}(E_{|D})$ *norm, i.e.* $T_Y f \in (S_P^{p,2}(D))^\perp$.

PROOF. It follows from Proposition 3.2.9 that

$$H_p^P(T_Y f, v) = \int_D (f, Pv)_x dx = 0,$$

if $v \in S_P^{p,2}(D)$. $\square$

We also have

PROPOSITION 3.3.11. *Let* $f \in L^2(F_{|D})$. *Then a necessary and sufficient condition for the convergence of the series* R_Y *in* $W^{p,2}(E_{|D})$ *is the convergence of the series*

$$\sum_{\mu=0}^{\infty} \|\mathcal{G}_Y^\mu T_Y f\|^2_{W^{p,2}(E_{|D})}. \tag{3.3.6}$$

PROOF. Since the scalar product $H_p^P(.,.)$ is equivalent to the usual one in $W^{p,2}(E_{|D})$ the convergence of the series (3.3.6) is equivalent to that of the series

$$\sum_{\mu=0}^{\infty} H_p^P(\mathcal{G}_Y^\mu T_Y f, \mathcal{G}_Y^\mu T_Y f).$$

Then the statement follows because $\mathcal{G}_Y$ is non negative and self-adjoint with respect to the scalar product $H_p^P(\cdot,\cdot)$. $\square$

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§3.4. On the Poincarè Lemma for elliptic differential complexes

We investigate now conditions for the vanishing of the cohomology groups

$$H(W^{r,2}(F_{|D})) = S_{P1}^{r,2}(D)/\mathfrak{N}_p^{r,2}(D)$$

of the complex (3.3.4).

From Theorem 3.3.7 and Proposition 3.3.6 of the previous section we have

Corollary 3.4.1. *$H(W^{r,2}(F_{|D})) = 0$ ($r \geq 0$) if and only if*

(1) $S^{r,2}_{P^1}(D) \subset domR^{p,r}_Y$;
(2) $\mathfrak{H}^{r,2}(D) = 0$.

Remark 3.4.2. We note that conditions (1) and (2) of Corollary 3.4.1 are applied not only to the domain D but also to the compatibility operator P^1. Indeed, the ellipticity of the complex does not guarantee that the P^1 is the right compatibility operator. For example, the complex

$$0 \to C^\infty(\Lambda^0) \xrightarrow{d^0} C^\infty(\Lambda^2) \xrightarrow{\Delta_2 d^1} C^\infty(\Lambda^1) \to 0$$

in $\mathbb{R}^2$, with $d^0 = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$, $d^1 = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$, and the Laplace operator Δ_2, is elliptic. However the Poincarè Lemma is not valid for this complex.

Let us clarify the conditions in Corollary 3.4.1.

Proposition 3.4.3. *$S^{r,2}_{P^1}(D) \subset domR^{p,r}_Y$ if and only if the natural map $i : \mathfrak{H}^{r,2}(D) \to H(W^{r,2}(F_{|D}))$ is bijective.*

Proof. It follows from statement (4) of Proposition 3.3.6, that the natural map $i : \ker T_Y \cap S^{r,2}_{P^1}(D) \to H(W^{r,2}(F_{|D}))$ is always injective.

Assume now that $S^{r,2}_{P^1}(D) \subset dom_r R_Y$. Then formula (3.3.3) and Proposition 3.3.4 imply that, for every $f \in S^{r,2}_{P^1}(D)$, the section $(f - PR_Y f)$ belongs to $\ker T_Y \cap S^{r,2}_{P^1}(D)$. Obviously, $(f - PR_Y f)$ belongs to the same cohomology class as f. By Lemma 3.3.8, $\mathfrak{H}^{r,2}(D) = \ker T_Y \cap S^{r,2}_{P^1}(D)$. Then the map i is surjective and, due to Proposition 3.3.6, is also injective.

On the other hand, if the natural map $i : \mathfrak{H}^{r,2}(D) \to H(W^{r,2}(F_{|D}))$ is surjective then, again using Lemma 3.3.8, for every $f \in S^{r,2}_{P^1}(D)$, there exist sections $\widetilde{f} \in \ker T_Y \cap S^{r,2}_{P^1}(D)$ and $u \in W^{p,2}(E_{|D})$ such that $f = \widetilde{f} + Pu$. In particular, due to Proposition 3.3.4, we obtain that $R_Y f = R_Y(\widetilde{f} + Pu) = R_Y Pu$. Now using Corollary 3.1.3 we conclude that the series $R_Y Pu$ converges in the $W^{p,2}(E_{|D})$-norm. Hence $R_Y(\widetilde{f} + Pu)$ also converges in $W^{p,2}(E_{|D})$-norm. Therefore, since $PR_Y f = PR_Y Pu = Pu$ and $Pu = f - \widetilde{f} \in W^{r,2}(F_{|D})$, we obtain that $f \in domR^{p,r}_Y$.
□

The triviality of the cohomology group $H(W^{r,2}(F_{|D}))$, implies, in particular, that the range $Im(P^{p,r})$ of the map $P^{p,r} : W^{p,2}(E_{|D}) \to W^{r,2}(F_{|D})$ is closed in $W^{r,2}(F_{|D})$. In the following statement $\overline{Im(P^{p,r})}$ stands for the closure of the range $Im(P^{p,r})$ in $W^{r,2}(F_{|D})$.

Proposition 3.4.4. *The range $Im(P^{p,r})$ is closed if and only if $\overline{Im(P^{p,r})} \subset domR^{p,r}_Y$ ($r \geq 0$).*

Proof. Let $f \in domR^{p,r}_Y$. Then $f - PR_Y f$ belongs to $\ker R_Y$ by (3.3.3). Since $\ker R_Y = \ker T_Y$ by Proposition 3.3.4, we obtain that $domR^{p,r}_Y = \ker T_Y \oplus Im(P^{p,r})$. If we assume that $\overline{Im(P^{p,r})} \subset domR^{p,r}_Y$ we obtain a sum decomposition

$$\overline{Im(P^{p,r})} = (\ker T_Y \cap \overline{Im(P^{p,r})}) \oplus Im(P^{p,r}). \tag{3.4.1}$$

On the other hand, if $f \in (\ker T_Y \cap \overline{Im(P^{p,r})}$ then there exists a sequence $\{u_N\} \subset W^{p,2}(E_{|D})$ such that $\lim_{N\to\infty} Pu_N = f$ in the $L^2(F_{|D})$-norm. Hence, due to Proposition 3.2.9,

$$\|f\|^2_{L^2(F_{|D})} = \lim_{N\to\infty} \int_D (f, Pu_N)_x dx = \lim_{N\to\infty} H^P_p(T_Y f, u_N) = 0. \tag{3.4.2}$$

Therefore

$$(\ker T_Y \cap \overline{Im(P^{p,r})}) = 0$$

and, by (3.4.1), the range $Im(P^{p,r})$ is closed.

Conversely, by (3) in Proposition 3.3.6 we have $\overline{Im(P^{p,r})} = \mathfrak{N}^{r,2}_p(D) \subset domR^{p,r}_Y$ and therefore the conclusion is obviously necessary. $\square$

Decomposition (3.4.1) becomes clearer if we remember that $T_Y = P^\star$ where $P^\star : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is the adjoint (in the sense of Hilbert spaces) of the operator P with respect to the scalar product $H^P_p(.,.)$ in $W^{p,2}(E_{|D})$ and the standard one in $L^2(F_{|D})$ (see Remark 3.2.15).

Using the integrals T_Y and $\mathcal{G}_Y$ we obtain simpler conditions for the first cohomology group of the complex (3.3.4) to be trivial in the case $r = 0$ and $m = p$. This is the case where solutions can be obtained with maximal global regularity. This applies for instance to the de Rham complex, but does not to the Dolbeault complex (see Example 3.6.4).

To simplify notations we will write $Im(P)$ instead of $Im(P^{p,0})$.

PROPOSITION 3.4.5. $H(L^2(F_{|D})) = 0$ *if and only if*

(1) *the range* $Im(P)$ *of the map* $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ *is closed in* $L^2(F_{|D})$*;*
(2) $\mathfrak{H}^{0,2}(D) = 0$.

PROOF. Necessity. Let $H(L^2(F_{|D})) = 0$ then $S^{0,2}_{P^1}(D) = Im(P)$. Hence, since $S^{0,2}_{P^1}(D)$ is a closed subspace of $L^2(F_{|D})$, $Im(P)$ is closed. The necessity of condition (2) of the theorem follows from Proposition 3.3.6.

Sufficiency. Let the range $Im(P)$ of the map $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ be closed in $L^2(F_{|D})$. Then the continuous map

$$P : (S^{p,2}_P(D))^\perp \to Im(P)$$

is one-to-one. Now, since $Im(P)$ and $(S^{p,2}_P(D))^\perp)$ are closed subspaces of $L^2(F_{|D})$ and $W^{p,2}(E_{|D})$ respectively, the open map theorem implies that there exists a positive constant c such that

$$\|v\|_{W^{p,2}(E_{|D})} \le c\|Pv\|_{L^2(F_{|D})}$$

for every $v \in (S^{p,2}_P(D))^\perp)$.

Therefore the Hermitian form

$$\widetilde{H}^P_p(u, v) = \int_D (Pu, Pv)_x dx$$

is a scalar product on $(S^{p,2}_P(D))^\perp$; and the topology induced in $(S^{p,2}_P(D))^\perp$ by this scalar product is equivalent to the original one.

Let f be a section in $S^{0,2}_{P^1}(D)$. Then the integral

$$\int_D (f, Pu)_x dx \ (v \in (S^{p,2}_P(D))^\perp))$$

defines a continuous linear functional on $(S^{p,2}_P(D))^\perp)$. Now, using Riesz representation theorem, we conclude that there exists $u \in (S^{p,2}_P(D))^\perp$ such that

$$\int_D (f, Pv)_x dx = \int_D (Pu, Pv)_x dx \tag{3.4.3}$$

for every $v \in (S^{p,2}_P(D))^\perp$. But then (3.4.3) holds for every $v \in W^{p,2}(E_{|D})$.

Furthermore, since for every $w \in C^\infty_0(F^*_{|D})$ the section $(*^{-1}_F w)$ belongs to $W^{p,2}(E_{|D})$, we have

$$\int_D < {}^tP^* w, f >_x dx = \int_D (f, P(*^{-1}_F w))_x dx = 0,$$

i.e. $P^*(f - Pu) = 0$ in D. Thus, since $f \in S^{0,2}_{P^1}(D)$, we conclude that $(f - Pu) \in kerP^1 \cap \ker P^* \cap L^2(F_{|D}) \subset C^\infty(F_{|D})$.

Finally, if we prove that the weak boundary values $({}^tC^*_j(f - Pu))_{|\partial D} = 0$ then $(f - Pu) \in \mathfrak{H}^{0,2}(D)$ and, due to condition (2) of the theorem, $Pu = f$ in D.

To this aim we fix a section $v^{(j)} \in C^\infty_{comp}(F^*_j)$ and find a section $v \in \mathcal{D}(E^*_{|\overline{D}})$ such that ${}^tB^*_j v = v^{(j)}$ on ∂D, and ${}^tB^*_i v = 0$ on ∂D, if $i \neq j$ (see Lemma 1.1.5). It is clear that $(*^{-1}_E v) \in W^{p,2}(E_{|D})$. Therefore, using (3.4.3) we obtain that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^tC^*_j(f - Pu) >_y ds =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \sum_{i=0}^{p-1} < {}^tB^*_i v, {}^tC^*_i(f - Pu) >_y ds =$$

$$= \int_D < {}^tP^* v, (f - Pu) >_y dy = \int_D ((f - Pu), P(*^{-1}_E v))_y dy = 0.$$

The proof of the theorem is complete. □

Corollary 3.4.6. *$H(L^2(F_{|D})) = 0$ if and only if*

(1) $\overline{Im(P)} \subset domR^{p,0}_Y$;
(2) $\mathfrak{H}^{0,2}(D) = 0$.

Proof. It follows form Propositions 3.4.4 and 3.4.5. □

Theorem 3.4.7. *The following conditions are equivalent:*

(1) $H(L^2(F_{|D})) = 0$;
(2) *there exists a constant $C > 0$ such that for every $g \in S^{0,2}_{P^1}(D)$*

$$\|g\|_{L^2(F_{|D})} \leq C\|T_Y g\|_{W^{p,2}(E_{|D})};$$

(3) *there exists a constant $C > 0$ such that for every $g \in S^{0,2}_{P^1}(D)$*

$$\|g\|_{L^2(F_{|D})} \leq C\|PT_Y g\|_{L^2(F_{|Y})}.$$

PROOF. Let $H(L^2(F_{|D})) = 0$. Then $S^{0,2}_{P^1}(D) = Im(P) = \overline{Im(P)}$. Hence, because $T_Y : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is the adjoint (in the sense of Hilbert spaces) of the operator P with respect to the scalar product $H^P_p(.,.)$ in $W^{p,2}(E_{|D})$ and the standard one in $L^2(F_{|D})$ (see Remark 3.15), the ranges of P and T_Y are closed (see, for example, [Hö1], Theorem 1.1.1), i.e. statement (2) holds.

If (2) holds then the range $Im(T_Y)$ is closed. Therefore, from Remark 3.2.15 and Theorem 1.1.1 of [Hö1], the range $Im(P)$ is closed. Moreover (2) and Lemma 3.3.8 imply that $\mathfrak{H}^{0,2}(D) = 0$, i.e., due to Proposition 3.4.5, condition (1) is satisfied.

Finally, Lemmata 3.2.7 and 3.2.5 imply that $S(T_Y g) = (T_Y g)^+$. In particular this means that

$$H^P_p(T_Y g, T_Y g) = \int_Y (PT_Y g, PT_Y g)_x dx = \|PT_Y g\|^2_{L^2(F_{|Y})}.$$

Therefore Proposition 3.2.12 implies that (2) and (3) are equivalent. □

EXAMPLE 3.4.8. Let $\{E^i, P^i\}$ be a short elliptic Hilbert complex of operators with constant coefficients:

$$0 \to C^\infty(E^\circ) \xrightarrow{P^\circ} C^\infty(E^1) \xrightarrow{P^1} C^\infty(E^2) \to 0$$

(see [T5], [AnNa]). Then, for any domaind D with connected boundary $\partial D \in C^\infty$, $\mathfrak{H}^{r,2}(D) = 0$ $(r \geq 0)$.

Indeed, since the complex is elliptic, P^1 is an operator with surjective symbol. Then, according to [T6], for a section $f \in S^{0,2}_{P^1}(D)$ and a convex domain G (with $D \Subset G \Subset \mathbb{R}^n$) one can find a sequence $\{f_N\} \subset C^\infty(E^1_{|G}) \cap S_{P^1}(G)$ such that $f_N \to f$ in the $L^2(E^1_{|D})$-norm. Because G is convex and $\{E^i, P^i\}$ is the complex of Hilbert, there exist sections $u_N \in C^\infty(E^0_{|G})$ with $Pu_N = f_N$ in G. Hence $f = \lim_{N\to\infty} Pu_N$ in the $L^2(E^1_{|D})$-norm. Now, arguing as in the proof of Proposition 3.4.4 (see (3.4.2)), we obtain $\mathfrak{H}^{0,2}(D) = 0$.

Since $\mathfrak{H}^{r,2}(D) \subset \mathfrak{H}^{0,2}(D)$, we conclude that $\mathfrak{H}^{r,2}(D) = 0$.

In particular, $f \in domR^{p,r}_Y \cap S_{P^1}(D)$ implies the solvability of Problem 3.3.1 (cf. [Rom2] for the Cauchy-Riemann system in $\mathbb{C}^2$).

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§3.5. Applications to a P- Neumann problem

In this section we show how Theorem 3.2.13 can be used to study a P- Neumann problem associated to elliptic differential operator $P \in do_p(E \to F)$ (see also §3.8).

As in §1.1, $\{B_j\}_{j=0}^{p-1}$ is a Dirichlet system of order $(p-1)$ on ∂D and $\{C_j\}_{j=0}^{p-1}$ the one which is associated to $\{B_j\}_{j=0}^{p-1}$ as in Lemma 1.1.6.

PROBLEM 3.5.1. *Let $r \geq 0$ and $\psi_j \in W^{r-p+j+1/2,2}(F_{j|\partial D})$ $(0 \leq j \leq p-1)$ be given sections. We want to find $\psi \in W^{p,2}(E_{|D})$ such that*

$$\begin{cases} P^*P\psi = 0 \text{ in } D, \\ {}^tC^*_j P\psi = \psi_j \text{ on } \partial D \ (0 \leq j \leq p-1), \\ (P\psi) \in W^{r,2}(F_{|D}). \end{cases}$$

The equation $P^*P\psi = 0$ in D has to be understood in the sense of distributions, while the boundary values are intended in the variational sense :
(3.5.1)

$$\int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) = \int_D (P\psi, Pv)_y dy \ for\ every\ v \in C^\infty(E_{|\overline{D}}).$$

In particular we obtain

Proposition 3.5.2. *A necessary condition in order that Problem 3.5.1 be solvable for given* $\psi_j \in W^{r-p+j+1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ *is that*

$$\int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) = 0 \ for\ every\ v \in S_P^{p,2}(D). \tag{3.5.2}$$

Proof. Indeed, because $C^\infty(E_{|\overline{D}})$ is dense in $W^{p,2}(E_{|D})$, formula (3.5.1) extends by continuity to $v \in W^{p,2}(E_{|D})$. □

Proposition 3.5.3. *Let* $\psi_j = 0$ $(0 \le j \le p-1)$. *Then* $\psi \in W^{p,2}(E_{|D})$ *is a solution of Problem 3.5.1 if and only if* $\psi \in S_P^{p,2}(D)$.

Proof. Obviously, a section $\psi \in S_P^{p,2}(D)$ is a solution of Problem 3.5.1 with $\psi_j = 0$ $(0 \le j \le p-1)$. Conversely, if ψ is a solution of Problem 3.5.1 with $\psi_j = 0$ $(0 \le j \le p-1)$ then $T_Y P\psi = 0$. Hence $\psi = \mathcal{G}_Y \psi = \lim_{\nu\to\infty} M^\nu \psi$, i.e. $\psi \in S_P^{p,2}(D)$.
□

The operator P^*P is a elliptic with C^∞ coefficients, and the ranks of the symbols of the boundary operators $({}^tC_j^*)$ are maximal in a neighbourhood of ∂D. Nevertheless, since, in general, the space $S_P^{p,2}(D)$ is not finite dimensional, Proposition 3.5.3 implies that the boundary value Problem 3.5.1 may be not elliptic.
In the following theorem we set

$$\widetilde{T_Y}(\oplus\psi_j) = \int_{\partial D} \sum_{j=0}^{p-1} < {}^tB_j^*(y)\Phi(x,y), \psi_j(y) >_y ds(y).$$

Theorem 3.5.4. *Problem 3.5.1 is solvable if and only if the series* $\sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j)$ *converges in the* $W^{p,2}(E_{|D})$*-norm and* $P\sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j) \in W^{r,2}(F_{|D})$.

Proof. Let Problem 3.5.1 be solvable and let $\psi \in W^{p,2}(E_{|D})$ be a solution. Then $\widetilde{T_Y}(\oplus\psi_j)) = T_Y P\psi$, and, due to Theorem 3.2.13, the series $R_Y P\psi = \sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j)$ converges in the $W^{p,2}(E_{|D})$-norm. Moreover, by Theorem 3.2.13, $P\sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j) = P\psi \in W^{r,2}(F_{|D})$.

Conversely, assume that the series $\sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j)$ converges in the $W^{p,2}(E_{|D})$-norm, and that $P\sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j) \in W^{r,2}(F_{|D})$. Let us we set $\psi = \sum_{\mu=0}^\infty \mathcal{G}_Y^\mu \widetilde{T_Y}(\oplus\psi_j)$. Then $P^*P\psi = 0$ in D. Hence to prove that ψ is a solution of Problem 3.5.1 we need only to prove only that ${}^tC_j^* P\psi = \psi_j$ on ∂D $(0 \le j \le p-1)$.

We note now that

$$\mathcal{G}_Y\psi = \mathcal{G}_Y \sum_{\mu=0}^{\infty} \mathcal{G}_Y^{\mu} \widetilde{T_Y}(\oplus\psi_j) = \sum_{\mu=0}^{\infty} \mathcal{G}_Y^{\mu} \widetilde{T_Y}(\oplus\psi_j) - \widetilde{T_Y}(\oplus\psi_j) = \psi - \widetilde{T_Y}(\oplus\psi_j).$$

Hence we obtain, using (3.1.2) and Stokes' formula :

$$\widetilde{T_Y}(\oplus\psi_j) = T_Y P\psi = \widetilde{T_Y}(\oplus {}^tC^* P\psi) \text{ in } Y.$$

Finally (1.3.5) implies that

$$(\psi_j - {}^tC_j^* P\psi)_{|\partial D} = ({}^tC_j^* P\widetilde{T_Y}(\oplus(\psi_j - {}^tC_j^* P\psi))^-)_{|\partial D} -$$
$$-({}^tC_j^* P\widetilde{T_Y}(\oplus(\psi_j - {}^tC_j^* P\psi))^+)_{|\partial D} = 0.$$

Theorem 3.5.4 is proved. □

Proposition 3.5.5. *Let* $\psi_j \in W^{-p+j+1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$. *If Problem 3.5.1 is solvable then the series*

$$\psi = \sum_{\nu=0}^{\infty} (\mathcal{G}_Y)^{\nu} \widetilde{T_Y}(\oplus\psi_j),$$

converging in the $W^{p,2}(E_{|D})$*-norm, is the (unique) solution of Problem 3.5.1 belonging to* $(S^{p,2}(D))^{\perp}$.

Proof. See the proof of Theorem 3.5.4. □

In the case where $P = \overline{\partial}$ (the Cauchy-Riemann system) in $\mathbb{C}^n$ such a formula was obtained by Kytmanov (see [Ky], p.177). For the matrix factorizations of the Laplace operator see §3.8.

In the remaining part of this section we will show how the P-Neumann problem 3.5.1 connects to the solvability of the equation $Pu = f$ and to the closedness of the image of the operator P.

Let us first investigate criterions for $f \in dom R_Y^{p,r}$ (see Theorem 3.3.7). To this purpose we consider the following problem.

Problem 3.5.6. *Given a section* $v \in (S^{p,2}(D))^{\perp} \cap W^{r+p,2}(E_{|D})$, *find a section* $\varphi \in W^{p,2}(E_{|D})$ *such that*

$$\begin{cases} T_Y P\varphi = v, \\ (P\varphi) \in W^{r,2}(F_{|D}). \end{cases}$$

Theorem 3.5.7. *Let* $f \in W^{r,2}(F_{|D})$ $(r \ge 0)$. *The following conditions are equivalent:*

(1) $f \in dom R_Y^{p,r}$;
(2) *for every* $v \in S_{\Delta}^{p,2}(D)$ *we have*

$$\begin{cases} \int_{-0}^{1} \frac{d_\lambda(H_p^P(E_\lambda T_Y f, v))}{1-\lambda} < \infty, \\ P(\int_{-0}^{1} \frac{dE_\lambda(T_Y f)}{1-\lambda}) \in W^{r,2}(F_{|D}); \end{cases}$$

(3) *The* P*-Neumann Problem 3.5.1 is solvable for* $\{\psi_j = ({}^tC_j^* P(T_Y f)^+)_{|\partial D}\}_{(0\le j\le p-1)}$;
(4) *Problem 3.5.6 is solvable for* $v = T_Y f$.

PROOF. (1) $\Leftrightarrow$ (2). The statement follows from the following chain of equalities:

$$\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}(T_Y f)=\lim_{\nu\to\infty}\sum_{\mu=0}^{\nu-1}\int_{-0}^{1}\lambda^{\mu}dE_{\lambda}(T_Y f)=$$

$$=\lim_{\nu\to\infty}\int_{-0}^{1}\sum_{\mu=0}^{\nu-1}\lambda^{\mu}dE_{\lambda}(T_Y f)=\lim_{\nu\to\infty}\int_{-0}^{1}\frac{(1-\lambda^{\nu})dE_{\lambda}(T_Y f)}{1-\lambda}.$$

(1) $\Leftrightarrow$ (3). Lemma 3.2.10 and Theorem 3.2.13 imply that

$$R_Y f=\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}T_Y f=T_Y P T_Y f+\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}T_Y P(S(T_Y f))=$$

$$=T_Y P T_Y f+\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}T_Y P((T_Y f)^+)=T_Y P T_Y f+\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}\widetilde{T_Y}(\oplus {}^tC_j^* P(T_Y f)^+).$$

But this means that the series $\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}\widetilde{T_Y}(\oplus {}^tC_j^* P(T_Y f)^+)$ converges in the $W^{p,2}(E_{|D})$-norm, and $P\sum_{\mu=0}^{\infty}\mathcal{G}_Y^{\mu}\widetilde{T_Y}(\oplus {}^tC_j^* P(T_Y f)^+)\in W^{r,2}(F_{|D})$ if and only if $f\in domR_Y^{p;r}$. Therefore the statement follows from Theorem 3.5.4.

(1) $\Leftrightarrow$ (4). Let $f\in domR_Y^{p;r}$ then (3.3.3) implies that $(f-PR_Y f)\in kerT_Y\cap W^{r,2}(F_{|D})$, that is $\varphi=R_Y f$, because $\ker T_Y=\ker R_Y^{p;r}$ by Proposition 3.3.4.

Conversely, if (4) holds then Theorem 3.2.13 implies that the series $R_Y P\varphi=R_Y f$ converges in the $W^{p,2}(E_{|D})$-norm, and $PR_Y f=PR_Y P\varphi=P\varphi\in W^{r,2}(F_{|D})$. Therefore $f\in domR_Y^{p;r}$.

The proof of Theorem 3.5.7 is complete. $\square$

Remark 3.5.8. We emphasize that the Neumann Problem 3.5.1 is the P-Neumann problem associated with the differential complex $\{E^i,P^i\}$ (see, for example, [T5], p. 136) at step $i=0$. However, as a rule, in order to solve the equation $Pu=f$, the P-Neumann problem was studied in the case $i=1$.

PROPOSITION 3.5.9. *Let $u\in S^{p,2}(D)^{\perp}$ and $\psi_j=({}^tC_j^* PS(u))_{|\partial D}$ $(0\le j\le p-1)$. Then the necessary condition (3.5.2) for the solvability of Problem 3.5.1 holds, i.e. for every $v\in S_P^{p,2}(D)$ we have*

$$\int_{\partial D}\sum_{j=0}^{p-1}<(*_{F_j}B_j v,\psi_j>_y ds(y)=0.$$

PROOF. Indeed, formula (3.5) implies that

$$\int_{\partial D}\sum_{j=0}^{p-1}<*_{F_j}B_j v,\psi_j>_y ds(y)=\int_{\partial D}\sum_{j=0}^{p-1}<*_{F_j}B_j v,{}^tC_j^* PS(u))>_y ds(y)=$$

$$=-\int_{Y\setminus D}(PS(u),PS(v))_y dy=-H_p^P(u,v)=0$$

for every $v\in S_P^{p,2}(D)$. $\square$

Because $S(T_Y f)=(T_Y f)^+$, Propositions 3.3.10 and 3.5.9 imply that condition (3.5.2) holds for $\psi_j=({}^tC_j^* P(T_Y f)^+)_{\partial D}$ $(0\le j\le p-1)$.

Now let us see the connection between the P-Neumann problem and the closedness of the range of the operator P.

PROPOSITION 3.5.10. $\overline{Im(T_Y)} = \overline{Im(T_Y P)} = (S_P^{p,2}(D))^{\perp}$.

PROOF. Proposition 3.3.10 and Lemma 3.2.7 imply that $Im(T_Y) \subset (S_P^{p,2}(D))^{\perp}$. Because $(S_P^{p,2}(D))^{\perp}$ is a closed subspace of $W^{p,2}(E_{|D})$, $\overline{Im(T_Y P)} \subset \overline{Im(T_Y)} \subset (S_P^{p,2}(D))^{\perp}$.

Conversely, formula (3.1.2) and Corollary 3.1.3 imply that

$$v = \sum_{\nu=0}^{\infty} \mathcal{G}_Y^{\nu} T_Y P v = \sum_{\nu=0}^{\infty} (Id - T_Y P)^{\nu} T_Y P v$$

for every $v \in (S_P^{p,2}(D))^{\perp})$. Therefore $(S_P^{p,2}(D))^{\perp} \subset \overline{Im(T_Y P)} \subset \overline{Im(T_Y)}$. $\square$

PROPOSITION 3.5.11. *The range $Im(T_Y)$ is closed if and only if the range $Im(T_Y P)$ is closed.*

PROOF. Let $Im(T_Y P)$ be closed. Then, due to Proposition 3.5.10,

$$(S_P^{p,2}(D))^{\perp} = \overline{Im(T_Y P)} = Im(T_Y P) \subset Im(T_Y) \subset (S_P^{p,2}(D))^{\perp}.$$

Hence the inclusions are equivalent and the range $Im(T_Y)$ is closed.

Conversely, if the range $Im(T_Y)$ is closed then Proposition 3.4.7 and Theorem 1.1.1 of [Hö1] imply that the range $Im(P)$ is closed. Therefore $Im(T_Y P) = \overline{Im(T_Y P)}$ because T_Y is in this case a topological homomorphism. $\square$

PROPOSITION 3.5.12. *The range $Im(P)$ is closed if and only if the P-Neumann Problem 3.5.1 is solvable for all $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$ satisfying (3.5.2).*

PROOF. Let $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$. Then $\widetilde{T}_Y(\oplus\psi_j) \in W^{p,2}(E_{|D})$ (see book [ReZs], 2.3.2.4). Moreover, if $\oplus\psi_j$ satisfies (3.5.2) then, due to (1.3.5), we have

$$H_p^P(\widetilde{T}_Y(\oplus\psi_i), v) = \int_{\partial D} \sum_{i=0}^{p-1} < *_{F_i} B_i v, {}^tC_i^* P \widetilde{T}_Y(\oplus\psi_i)^- - {}^tC_i^* P \widetilde{T}_Y(\oplus\psi_i)^+ >_y ds =$$

$$\int_{\partial D} \sum_{i=0}^{p-1} < *_{F_i} B_i v, \psi_i >_y ds = 0$$

for every $v \in S_P^{p,2}(D)$. That is, $\widetilde{T}_Y(\oplus\psi_j) \in (S_P^{p,2}(D))^{\perp}$. On the other hand, if $Im(P)$ is closed then, according to Propositions 3.4.7, 3.5.10, and 3.5.11, and Theorem 1.1.1 of [Hö1] $Im(T_Y P) = (S_P^{p,2}(D))^{\perp}$. In particular it means that there exists a section $\varphi \in W^{p,2}(E_{|D})$ such that $T_Y P\varphi = \widetilde{T}_Y(\oplus\psi_j)$. Therefore, from Theorem 3.2.13 and Corollary 3.1.3, the series

$$\sum_{\nu=0}^{\infty} \mathcal{G}_Y^{\nu} T_Y P\varphi = \sum_{\nu=0}^{\infty} \mathcal{G}_Y^{\nu} \widetilde{T}_Y(\oplus\psi_j)$$

converges in the $W^{p,2}(E_{|D})$-norm. Now using Theorem 3.5.4 we conclude that Problem 3.5.1 is solvable.

Conversely, let $v \in (S_P^{p,2}(D))^{\perp}$. Then $({}^tC_j^* PS(v))_{|\partial D} \in W^{-p+j+1/2,2}(F_{j|\partial D})$, and, due to Proposition 3.5.9, $(\oplus {}^tC_j^* PS(v))_{|\partial D}$ satisfies (3.5.2). Hence, if Problem

3.5.1 is solvable for all $\oplus\psi_j$ satisfying (3.5.2), there exists a section $\psi \in S^{p,2}_{\Delta}(D)$ such that $(\oplus^t C_j^* P\psi)_{|\partial D} = (\oplus^t C_j^* PS(v))_{|\partial D}$. In particular, from Lemma 3.2.10, we have

$$v = T_Y P v + T_Y PS(v) = T_Y P v + \widetilde{T_Y}(\oplus^t C_j^* PS(v)) = T_Y P(v + \psi).$$

Therefore $Im(T_Y P) = (S^{p,2}_P(D))^{\perp}$, i.e. $Im T_Y P$ is closed, and, due to Propositions 3.4.7, 3.5.11 and Theorem 1.1.1 of [Hö1], $Im(P)$ is closed. □

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§3.6. Examples of applications to P-Neumann problems

Using Proposition 3.5.12 we can obtain a result on the solvability of the P-Neumann Problem 3.5.1 in the case where P is determined elliptic.

COROLLARY 3.6.1. *Let P be a determined elliptic operator in X such that the operators P and P^*P have bilateral fundamental solutions on X. Then Problem 3.5.1 is solvable for every $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$ satisfying (3.5.2).*

PROOF. According to Corollary 1.1.9, for every $f \in L^2(F_{|D})$ there exist a $W^{p,2}(E_{|D})$ solution of the equation $Pu = f$. In particular, $Im((P)$ is closed, and therefore, the statement follows from Proposition 3.5.12. □

We note that in Corollary 3.6.1 we obtain maximal Sobolev regularity for the solutions of the boundary value Problem 3.5.1. However the nullspace of the problem may be not finite dimensional (see Proposition 3.5.3) and hence this may be not an elliptic boundary value problem.

EXAMPLE 3.6.2. Let $P = \Delta$ be Laplace operator in $\mathbb{R}^n$. Then $P^*P = \Delta^2$ and hence the operators P and P^*P have bilateral fundamental solutions in X.

Let $D \Subset \mathbb{R}^n$ be a domain with C^∞-smooth boundary ∂D. As a Dirichlet system on ∂D we can take the system $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$. Then, by simple calculations, the system $\{C_0 = -\frac{\partial}{\partial n}, C_1 = 1\}$ is the system associated to $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$ in Lemma 1.1.6 Therefore Corollary 3.6.1 implies that the problem

$$\begin{cases} \Delta^2\psi = 0 \ in \ D, \\ -\frac{\partial}{\partial n}\Delta\psi = \psi_0 \ on \ \partial D, \\ \Delta\psi = \psi_1 \ on \ \partial D, \\ \psi \in W^{2,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-3/2,2}(\partial D)$, $\psi_1 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D} \left(\psi_0(y)\overline{v(y)} - \psi_1(y)\overline{\frac{\partial v}{\partial n}(y)} \right) ds(y) = 0 \ for \ every \ harmonic \ W^{2,2}(D){-}function \ v.$$

EXAMPLE 3.6.3. Let P be the Cauchy - Riemann system on the plane $\mathbb{C}^1 \cong \mathbb{R}^2$, i.e. $P = \frac{\partial}{\partial x_1}+\sqrt{-1}\frac{\partial}{\partial x_2}$. In the complex form with $z = x_1+\sqrt{-1}x_2$, $\overline{z} = x_1-\sqrt{-1}x_2$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2})$, we have $P = 2\frac{\partial}{\partial \overline{z}}$, $P^* = -2\frac{\partial}{\partial z}$. Then $P^*P = -\Delta$ is the Laplace operator in $\mathbb{R}^2$ and hence the operators P and P^*P have bilateral fundamental solutions on X.

Let $D \Subset \mathbb{R}^2$ be a domain with C^∞-smooth boundary ∂D. As a Dirichlet system on ∂D we can take the system $\{B_0 = 1\}$. Then, setting

$$\begin{cases} \rho(x) = -dist(x, \partial D), \ x \in \overline{D}, \\ \rho(x) = dist(x, \partial D), \ x \notin \overline{D}, \end{cases}$$

the function ρ belongs to the class of functions defining the domain D ($D = \{x \in X : \rho(x) < 0\}$), $|d\rho| = \sqrt{\sum_{j=1}^{2}(\frac{\partial \rho}{\partial x_j})^2} = 1$ in a neighbourhood of ∂D and the system $\{C_0 = 2\frac{\partial \rho}{\partial \overline{z}}\}$ is the system associated to $\{B_0 = 1\}$ in Lemma 1.1.6. Therefore Corollary 3.6.1 implies that the problem

$$\begin{cases} -\Delta\psi = 0 \ in \ D, \\ 4\frac{\partial \rho}{\partial z}\frac{\partial \psi}{\partial \overline{z}} = \psi_0 \ on \ \partial D, \\ \psi \in W^{1,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-1/2,2}(\partial D)$, satisfying

$$\int_{\partial D} \psi_0(y)\overline{v(y)}ds(y) = 0 \ for \ every \ holomorphic \ W^{1,2}(D) - function \ v.$$

The problem above is nothing but the $\overline{\partial}$ -Neumann problem for functions in $\mathbb{C}^1$.

Consider now situation where the operator P is overdetermined (elliptic).

EXAMPLE 3.6.4. Let P be the Cauchy - Riemann system in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ ($n > 1$), i.e. $P = \begin{pmatrix} \frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_{n+1}} \\ \cdots \\ \frac{\partial}{\partial x_n} + \sqrt{-1}\frac{\partial}{\partial x_{2n}} \end{pmatrix}$. In the complex form with $z_j = x_j + \sqrt{-1}x_{n+j}$, $\overline{z}_j = x_j - \sqrt{-1}x_{n+j}$, $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial x_{n+j}})$, $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial x_{n+j}})$, we have $P = 2\begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \cdots \\ \frac{\partial}{\partial \overline{z}_n} \end{pmatrix}$ $(= 2\overline{\partial})$, $P^* = -2\begin{pmatrix} \frac{\partial}{\partial z_1} \\ \cdots \\ \frac{\partial}{\partial z_n} \end{pmatrix}$ $(= 2\partial)$. Then $P^*P = -\Delta$ is the Laplace operator in $\mathbb{R}^{2n}$ and hence the operator P^*P has a bilateral fundamental solution in X. However, due to the removability theorem for compact singularities of holomorphic functions in $\mathbb{C}^n$, the Cauchy-Riemann system in $\mathbb{C}^n$ has no right fundamental solution.

It is known that if the domain D is not pseudo-convex then the range $Im(P)$: $W^{1,2}(D) \to [L^2(D)]^n$ may be not closed. But even in a strictly convex domain D we can not achieve maximal global regularity for solutions of the equation $\overline{\partial}u = f \in [L^2(D)]^n$.

Indeed, let D be the ball $B(0, R)$ in $\mathbb{C}^2$ with centre at 0 and radius $0 < R < \infty$. Then $f = \begin{pmatrix} 0 \\ \frac{1}{R-z_1} \end{pmatrix} \in [L^2(D)]^2$ and the function $u = \frac{\overline{z}_2}{R-z_1} \in L^2(D)$ is a solution of the equation $\overline{\partial}u = f$ in D. Because

$$\frac{\partial u}{\partial z_1} = \frac{\overline{z}_2}{(R - z_1)^2} \notin L^2(D)$$

we conclude that $u \notin W^{1,2}(D)$.

Assume that there exists a function $v \in W^{1,2}(D)$ satisfying $\overline{\partial} v = f$. Then $v = u + h$ where h is a holomorphic L^2-function in the ball D and $\frac{\partial v}{\partial z_1} \in L^2(D)$. Hence

$$\|\frac{\partial v}{\partial z_1}\|^2_{L^2(D)} = lim_{\varepsilon\to 0}\|\frac{\partial v}{\partial z_1}\|^2_{L^2(D_\varepsilon)} = lim_{\varepsilon\to 0}(\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D_\varepsilon)} + \|\frac{\partial h}{\partial z_1}\|^2_{L^2(D_\varepsilon)}+$$

$$-\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z} - \overline{\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z}}) < \infty.$$

On the other hand,

$$-\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z} =$$

$$= \frac{1}{2\sqrt{-1}}\int_{|z_1|\le R-\varepsilon}\int_{r=0}^{(R-\varepsilon)^2-|z_1|^2}\int_{|z_2|=r}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})\frac{\sqrt{-1}d\overline{z}_2}{\overline{z}_2}rdrdz_1\wedge d\overline{z}_1 = 0$$

because $\frac{1}{\overline{z}_2}\frac{\partial u}{\partial z_1}$, $(\overline{\frac{\partial h}{\partial z_1}})$ are anti-holomorphic with respect to z_2 and hence

$$\int_{|z_2|=r}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})\frac{d\overline{z}_2}{\overline{z}_2} = 0 \ (0 < r < R).$$

Therefore we obtain

$$\|\frac{\partial v}{\partial z_1}\|_{L^2(D)} = lim_{\varepsilon\to 0}(\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D_\varepsilon)} + \|\frac{\partial h}{\partial z_1}\|^2_{L^2(D_\varepsilon)}) < \infty$$

contradicting $\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D)} = \infty$.

Thus we proved that for every ball $D = B(0,R) \subset \mathbb{C}^2$ there exists a closed differential $(0,1)$-form f with coefficients in $L^2(D)$ for which there is no $W^{1,2}(D)$-solution of the equation $\overline{\partial} u = f$ (cf. s. 3.8.2 below and [Sh6] for th Sobolev spaces and [Ke] for an analogous result for Hölder spaces).

Now using results of [Kohn] (on triviality of the "harmonic" spaces $\widetilde{\mathfrak{H}}^{0,2}(D)$), Proposition 3.5.13 and Lemma 3.5.5 we conclude that the image $Im(\overline{\partial}) : W^{1,2}(D) \to [L^2(D)]^2$ is not closed.

Let ρ be as in Example 3.6.3 then ρ belongs to the class of functions defining the domain D ($D = \{x \in X : \rho(x) < 0\}$), $|d\rho| = \sqrt{\sum_{j=1}^{2n}(\frac{\partial\rho}{\partial x_j})^2} = 1$ in a neighbourhood of $\partial D \in C^\infty$ and the system $\{C_0 = 2(\frac{\partial\rho}{\partial\overline{z}_1},\ldots,\frac{\partial\rho}{\partial\overline{z}_n})\}$ is the system associated to $\{B_0 = 1\}$ in Lemma 1.1.6.

Therefore, even if D is a ball, the boundary value problem

$$\begin{cases} -\Delta\psi = 0 \ in \ D, \\ 4\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}\frac{\partial\psi}{\partial\overline{z}_j} = \psi_0 \ on \ \partial D, \end{cases}$$

is not solvable in $W^{1,2}(D)$ for all (complex valued) data $\psi_0 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D}\psi_0(y)\overline{v(y)}ds(y) = 0 \ for \ every \ holomorphic \ W^{1,2}(D) - function \ v.$$

The problem above is nothing but the $\overline{\partial}$ -Neumann problem for functions in $\mathbb{C}^n$. Results about the solvability of this problem could be found, for example, in [Ky].

It is easier to prove that we can not achieve the maximal global regularity in the case where boundary of D is more "flat". For instance, if D is the bidisk in $\mathbb{C}^2$ with centre at 0 and radius $0 < R < \infty$, then arguing as before one sees that for $f = \begin{pmatrix} 0 \\ \frac{1}{(R-z_1)^\delta} \end{pmatrix} \in [L^2(D)]^2$ $(1/2 < \delta < 1)$ there is no $W^{1,2}(D)$-solution of the equation $\overline{\partial} u = f$ in D.

Example 3.6.5. Let $X = \mathbb{R}^n$ and $P = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \\ \cdots \\ \frac{\partial^2}{\partial x_n^2} \end{pmatrix}$. Then $P^*P = -\sum_{j=1}^n \frac{\partial^4}{\partial x_j^4}$. It is clear that P^*P has a bilateral fundamental solution on X but the operator P has only a left one.

However, it is not difficult to see that in every domain D, where we can find a solution with maximal (global) regularity of the equation $grad(u) = f$ in D, we can also solve with maximal (global) regularity the equation $Pu = f$. For instance, we can do it in every convex domain with $\partial D \in C^2$.

As a Dirichlet system on ∂D we can take the system $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$. If the function ρ is as in Example 3.6.3, then ρ belongs to the class of functions defining the domain D $(D = \{x \in X : \rho(x) < 0\})$, $|d\rho| = \sqrt{\sum_{j=1}^n (\frac{\partial \rho}{\partial x_j})^2} = 1$ in a neighbourhood of ∂D and the system of boundary differential operators

$$\{C_0 = -\left(\frac{\partial \rho}{\partial x_1}\frac{\partial}{\partial x_1}, \dots, \frac{\partial \rho}{\partial x_n}\frac{\partial}{\partial x_n}\right), \qquad C_1 = \left(\left(\frac{\partial \rho}{\partial x_1}\right)^2, \dots, \left(\frac{\partial \rho}{\partial x_n}\right)^2)\right)\}$$

is the system associated to $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$ in Lemma 1.1.6.

Therefore Proposition 3.5.12 implies that the problem

$$\begin{cases} \sum_{j=1}^n \frac{\partial^4}{\partial x_j^4}\psi = 0 \ in\ D, \\ -\sum_{j=1}^n \frac{\partial \rho}{\partial x_j}\frac{\partial^3 \psi}{\partial x_j^3} = \psi_0 \ on\ \partial D, \\ \sum_{j=1}^n (\frac{\partial \rho}{\partial x_j})^2 \frac{\partial^2 \psi}{\partial x_j^2} = \psi_1 \ on\ \partial D, \\ \psi \in W^{2,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-3/2,2}(\partial D)$, $\psi_1 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D} \left(\psi_0(y)\overline{v(y)} - \psi_1(y)\overline{\frac{\partial v}{\partial n}(y)}\right) ds(y) = 0 \ for\ evey\ S_P^{2,2}(D) - function\ v$$

in every convex domain D with a C^∞-smooth boundary ∂D.

Obviously, $S_P^{2,2}(D)$ consists of all polynomials of the form

$$\sum_{k \neq i} a_{k,i} x_k x_i + \sum_{j=1}^n b_j x_j + c$$

where $a_{k,i}, b_j, c \in \mathbb{C}^1$.

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§3.7. Applications to the Cauchy and Dirichlet problems

We have proved the solvability of the Dirichlet problem for an determined elliptic operator $P^*P = \Delta \in do_{2p}(E \to E)$ in Lemma 3.2.2. Let us now obtain a formula for the solution of this problem. In the following proposition $\mathcal{G}_Y(\oplus\psi_j)$ stands for the integral

$$\mathcal{G}_Y(\oplus\psi_j)(x) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j(y)^t P^*(y)\Phi_Y(x,y), \psi_j >_y ds.$$

Proposition 3.7.1. *Let the operator P satisfy the Uniqueness Condition in the small on X and ∂D be connected. Then, if $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1, m \ge p)$, the series*

$$\psi = \sum_{\nu=0}^{\infty} (T_Y P)^\nu \mathcal{G}_Y(\oplus\psi_j),$$

converging in the $W^{p,2}(E_{|D})$-norm, is the (unique) $W^{m,2}(E_{|D})$-solution of the Dirichlet problem for the operator P^*P and the data ψ_j $(0 \le j \le p-1)$.

Proof. We proved in Lemma 3.2.2 that for sections $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ there exists a unique solution $\psi \in W^{m,2}(E_{|D})$ of the Dirichlet problem. Theorem 3.2.13 and Corollary 3.1.3 imply that

$$\psi = \lim_{\nu\to\infty} (T_Y P)^\nu \psi + \sum_{\nu=0}^{\infty} (T_Y P)^\nu M\psi =$$

$$= \lim_{\nu\to\infty} (T_Y P)^\nu \psi + \sum_{\nu=0}^{\infty} (T_Y P)^\nu \mathcal{G}_Y(\oplus\psi_j).$$

On the other hand, under the hypothesis of the proposition $\widetilde{S}^{p,2}_P(D) = W^{p,2}_0(E_{|D})$ (see Remark 3.2.14), and therefore $\lim_{\nu\to\infty}(T_Y P)^\nu\psi = 0$, i.e.

$$\psi = \sum_{\nu=0}^{\infty} (T_Y P)^\nu \mathcal{G}_Y(\oplus\psi_j),$$

which was to be proved. □

This formula may be useful in cases where the Green function is known for a large domain Y (for instance where Y is a ball in $\mathbb{R}^n$, Δ is the usual Laplace operator and $D \Subset Y$ is a domain with connected boundary, for which the Green function is not known).

We consider now the degenerate Cauchy problem for the operator P with the Cauchy data given on the whole boundary (i.e. $S = \partial D$) (cf. Problem 2.4.1 and Theorem 2.4.2).

Problem 3.7.2. *Let $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1, m \ge p)$ be given sections. It is required to find a section $\psi \in W^{m,2}(E_{|D})$ such that*

$$\begin{cases} P\psi = 0 \text{ in } D, \\ B_j\psi = \psi_j \text{ on } \partial D, \ (0 \le j \le p-1). \end{cases}$$

Proposition 3.7.3. *Let $u \in W^{m,2}(E_{|D})$. The following conditions are equivalent:*

(1) $u \in S_P^{m,2}(D)$;
(2) $\mathcal{G}_Y u = u$ *in* D;
(3) $T_Y P u = 0$ *in* D.

Proof. Formula (3.1.2) implies that (2) and (3) are equivalent. Let $\mathcal{G}_Y u = u$ in D then, due to Theorem 3.2.13 $u = (\lim_{\nu\to\infty} \mathcal{G}_Y^\nu) \in S_P^{p,2}(D)$. Since $u \in W^{m,2}(E_{|D})$ we conclude that $u \in S_P^{m,2}(D)$. □

Proposition 3.7.4. *Let $u \in S_\Delta^{m,2}(D)$. The following conditions are equivalent:*

(1) $u \in S_P^{m,2}(D)$;
(2) $\mathcal{G}_Y u = 0$ *in* $Y \backslash \overline{D}$;
(3) $T_Y P u = 0$ *in* $Y \backslash \overline{D}$.

Proof. Formula (3.1.2) implies that (2) and (3) are equivalent. Let $T_Y P u = 0$ in $Y \backslash \overline{D}$ then, $T P_Y u = 0$ in D and the statement follows from Proposition 3.7.3. □

Corollary 3.7.5. *Let $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$. Then Problem 3.7.2 is solvable if and only if $\mathcal{G}_Y(\oplus\psi_j) = 0$ in $Y \backslash \overline{D}$.*

Proof. If Problem 3.7.2 is solvable and $\psi \in S_P^{m,2}(D)$ is the solution then $\mathcal{G}_Y(\oplus\psi_j) = \mathcal{G}_Y \psi$. Using Theorem 1.1.7 we conclude that $\mathcal{G}_Y(\oplus\psi_j) = 0$ in $Y \backslash \overline{D}$.

Conversely, if $\mathcal{G}_Y(\oplus\psi_j) = 0$ in $Y \backslash \overline{D}$ then (1.3.5) mplies that

$$(B_j \mathcal{G}_Y(\oplus\psi_j)^-)_{|\partial D} =$$

$$(B_j \mathcal{G}_Y(\oplus\psi_j)^-)_{|\partial D} - (B_j \mathcal{G}_Y(\oplus\psi_j)^+)_{|\partial D} = \psi_j \ (0 \le j \le p-1). \tag{3.7.1}$$

We set now $\psi = \mathcal{G}_Y(\oplus\psi_j)^-$. The Theorem on boundedness for potential (coboundary) operators in Sobolev spaces (see [ReSz], 2.3.2.5) implies that $\psi \in S_\Delta^{m,2}(D)$. On the other hand (3.7.1) implies that $\mathcal{G}_Y \psi = \mathcal{G}_Y(\oplus\psi_j)$, i.e. $\mathcal{G}_Y \psi = 0$ in $Y \backslash \overline{D}$. Therefore the statement follows from Proposition 3.7.4. □

For the Cauchy-Riemann system and the Martinely-Bochner integral Corollary 3.7.5 was obtained by Kytmanov (see [Ky], p. 170), and for matrix factorizations of the Laplace operator in $\mathbb{R}^n$ it was proved by one of the authors (see [Sh3]).

In [ShT2] (see also Theorem 2.4.2) necessary and sufficient conditions for the solvability of the Cauchy Problem 3.7.2 were obtained in terms of the Green operator $\mathcal{G}$ in the case where the coefficients of the operator P are real analytic or, if P is determined elliptic, where the Uniqueness Condition in the small on X holds for the operator P (see Remark 3.2.14). æ

§3.8. Examples for the matrix factorizations of the Laplace operator in $\mathbb{R}^n$

In this section we illustrate the theorem on iterations on the concrete examples (cf. [Sh3], [Sh6]). We consider the situation where $X = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{C}^k$, $F = \mathbb{R}^n \times \mathbb{C}^l$, $l \ge k$, and P is a matrix factorization of the Laplace operator in $\mathbb{R}^n$ (see Definition 2.14.1.2).

Throughout of this section $\Phi = \varphi_n I_k$, where φ_n is the standard fundamental solution of the convolution type of the Laplace operator in $\mathbb{R}^n$ and $\mathcal{G}$, T are the corresponding integrals with kernel Φ.

§3.8.1 The operator $\mathcal{G}$ in $W^{1,2}(E_{|D})$.

EXAMPLE 3.8.1.1. Let P is a matrix factorization of the Laplace operator in $\mathbb{R}^n$, $Y = X = \mathbb{R}^n$. Denoting by $\widetilde{S}^{1,2}_{\Delta}(\mathbb{R}^n \backslash \overline{D})$ the subspace of harmonic in $\mathbb{R}^n \backslash \overline{D}$ functions which are zero at infinity, and then arguing as in §3.2 we obtain (cf. [Sh3]) that

$$\lim_{\nu\to\infty} \mathcal{G}^\nu = \Pi(S^{p,2}_P(D)),$$

$$\lim_{\nu\to\infty} (TP)^\nu = \Pi(W^{1,2}_0(E_{|D})) + \Pi(\widetilde{S}^{p,2}_P(\mathbb{R}^n \backslash \overline{D})),$$

where $\widetilde{S}^{p,2}_P(\mathbb{R}^n \backslash \overline{D})$ is the space of functions satisfying $Pu = 0$ in $\mathbb{R}^n \backslash \overline{D}$ which are zero at infinity.

EXAMPLE 3.8.1.2. A.V. Romanov (see [Rom2]) obtained Theorem 3.2.13 for

$$P = 2\begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \cdots \\ \frac{\partial}{\partial \overline{z}_n} \end{pmatrix}$$

in $\mathbb{C}^n$ $(n \geq 2)$. He had $P^*P = -\Delta_{2n}$,

$$(\mathcal{G}u)(z) = (Mu)(z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_{\partial D} \sum_{j=1}^n (-1)^{j-1} \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} u(\zeta) d\overline{\zeta}[j] \wedge d\zeta,$$

$$(TPu)(z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int_D \sum_{j=1}^n \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) d\overline{\zeta} \wedge d\zeta$$

is the Martinelli-Bochner integral, where $z_j = x_j + \sqrt{-1}x_{j+n}$, $\zeta_j = y_j + \sqrt{-1}y_{j+n}$, $x, y \in \mathbb{R}^{2n}$, and $d\overline{\zeta}[j] = d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_{j-1} \wedge d\overline{\zeta}_{j+1} \ldots \wedge d\overline{\zeta}_n$. In this case, if ∂D is connected, the theorem on removable compact singularities of holomorphic functions implies that $S^{m,2}_P(\mathbb{C}^n \backslash \overline{D}) = \{0\}$ $(m \geq 1)$.

EXAMPLE 3.8.1.3. If P be the gradient operator in $\mathbb{R}^n$ $(n \geq 3)$ then $P^*P = -\Delta_n$,

$$\mathcal{G}u)(x) = \frac{1}{\sigma_n} \int_{\partial D} \sum_{j=1}^n (-1)^{j-1} \frac{y_j - x_j}{|y - x|^n} u(y) dy[j],$$

$$(TPu)(x) = \frac{1}{\sigma_n} \int_{\partial D} \sum_{j=1}^n \frac{y_j - x_j}{|y - x|^n} \frac{\partial u}{\partial y_j}(y) dy$$

where σ_n is the area of the unit sphere $\mathbb{R}^n$. In this case $S^{m,2}_P(D) = \mathbb{C}^1$, and, if ∂D is connected, $S^{m,2}_P(\mathbb{R}^n \backslash \overline{D}) = \{0\}$ $(m \geq 1)$.

EXAMPLE 3.8.1.4. Let $x \in \mathbb{R}^{4n}$ $(n \geq 1)$, $q_j = x_j + \sqrt{-1}x_{j+2n}$, $\frac{\partial}{\partial q_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial x_{j+n}})$ $\frac{\partial}{\partial \overline{q}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial x_{j+n}})$ $(1 \leq j \leq 2n)$, and let

$$Q_j = 2\begin{pmatrix} \frac{\partial}{\partial q_j} & \frac{\partial}{\partial q_{j+n}} \\ -\frac{\partial}{\partial \overline{q}_{j+n}} & \frac{\partial}{\partial \overline{q}_j} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 \\ \cdots \\ Q_n \end{pmatrix}.$$

Then $Q^*Q = -I_2\Delta_{4n}$ and, if $r_j = y_j + \sqrt{-1}y_{j+2n}$ $(1 \le j \le 2n)$, $y \in R^{4n}$, and

$$D(q,j) = \begin{pmatrix} (-1)^{2n+j-1} d\overline{q} \wedge dq[j] & (-1)^{n+j-1} d\overline{q}[j+n] \wedge dq \\ (-1)^{3n+j-1} d\overline{q} \wedge dq[j+n] & (-1)^{j-1} d\overline{q}[j] \wedge dq \end{pmatrix},$$

$$(\mathcal{G}u)(r) = \frac{(-1)^n(2n-1)!}{(2\pi)^{2n}} \sum_{j=1}^{n} D(q,j) \begin{pmatrix} q_j - r_j & q_{j+n} - r_{j+n} \\ -(\overline{q}_{j+n} - \overline{r}_{j+n}) & \overline{q}_j - \overline{r}_j \end{pmatrix} \frac{u(q)}{|q-r|^{4n}},$$

$$(TQu)(r) = \frac{(-1)^n(2n-1)!}{2(2\pi)^{2n}} \sum_{j=1}^{n} \begin{pmatrix} q_j - r_j & q_{j+n} - r_{j+n} \\ -(\overline{q}_{j+n} - \overline{r}_{j+n}) & \overline{q}_j - \overline{r}_j \end{pmatrix} \frac{(Q_j u)(q)}{|q-r|^{4n}} d\overline{q} \wedge dq.$$

Example 3.8.1.5. Let Λ^q be the bundle of (complex valued) exterior forms of degree q over $\mathbb{R}^n$ ($\Lambda^q \ne 0$ only for $0 \le q \le n$); let $d_q \in do_1(\Lambda^q \to \Lambda^{q+1})$ be the exterior derivative operator, and $d_q^* \in do_1(\Lambda^{q+1} \to \Lambda^q)$ be the formal adjoint operator of d_q. Then for the "laplacians" of the de Rham complex $(d_q^* d_q + d_{q-1} d_{q-1}^*) \in do_2(\Lambda^q \to \Lambda^q)$ we have $(d_q^* d_q + d_{q-1} d_{q-1}^*) = I_{i(q)}\Delta_n$ (see [T5], p.85). Therefore the operators $P_q = \begin{pmatrix} d_q \\ d_{q-1}^* \end{pmatrix} \in do_1(\Lambda^q \to (\Lambda^{q+1}, \Lambda^{q-1}))$ are matrix factorizations of the Laplace operator in $\mathbb{R}^n$. The space $S^{m,2}_{P_q^* P_q}(D)$ is the space of the differential forms of degree q whose coefficients are harmonic $W^{1,2}(D)$-functions.

Let $n = 3$ and $q = 1$, $D = B_1$ be the unit ball in $\mathbb{R}^3$. Then $l = 4$, $k = 3$ and

$$P_q = \begin{pmatrix} \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \end{pmatrix}.$$

It is easy to check that the vector $x \in S^{m,2}_{P_q^* P_q}(B_1)$ belongs to $S^{m,2}_{P_q}(\mathbb{R}^n \backslash B_1)$ with $S(u) = \frac{x}{|x|^3}$ and $m \le 0$. Hence $\ker \mathcal{G} = S^{m,2}_{P_q}(\mathbb{R}^n \backslash B_1) \ne 0$ in this case.

Example 3.8.1.6. Let $\Lambda^{t,q}$ be the bundle of (complex valued) exterior forms of bidegree (t,q) over $\mathbb{C}^n$, $\Lambda^{t,q} \ne 0$ only for $0 \le q \le n$, $0 \le t \le n$. Let $\overline{\partial}_{t,q} \in do_1(\Lambda^{t,q} \to \Lambda^{t,q+1})$ be the Cauchy- Riemann operator extended to forms of bidegree (t,q), and let $\overline{\partial}^*_{t,q} \in do_1(\Lambda^{t,q+1} \to \Lambda^{t,q})$ be the formal adjoint operator of $\overline{\partial}_{t,q}$. Then for the "laplacians" of the Dolbeault complex $(\overline{\partial}^*_{t,q}\overline{\partial}_{t,q} + \overline{\partial}_{t,q-1}\overline{\partial}^*_{t,q-1}) \in do_2(\Lambda^{t,q} \to \Lambda^{t,q})$ we have $c(t,q)(\overline{\partial}^*_{t,q} d_q + \overline{\partial}_{t,q-1}\overline{\partial}^*_{t,q-1}) = I_{i(t,q)}\Delta_{2n}$ (see [T5], p.88). Therefore the operators $P_{t,q} = \sqrt{c(t,q)} \begin{pmatrix} \overline{\partial}_{t,q} \\ \overline{\partial}^*_{t,q-1} \end{pmatrix} \in do_1(\Lambda^{t,q} \to (\Lambda^{t,q+1}, \Lambda^{t,q-1}))$ are matrix factorizations of the Laplace operator in $\mathbb{R}^{2n}$. The space $S^{m,2}_{P^*_{t,q}P_{t,q}}(D)$ is the space of the differential forms of bidegree (t,q) whose coefficients are harmonic $W^{1,2}(D)$-functions.

In Example 3.6.4 we have seen that we can not obtain solutions of $\overline{\partial}$-problem with maximal global regularity. Let us obtain a formula for solutions (as in §3.3) with loosing some global regularity.

In the following theorem D is a bounded domain in $\mathbb{C}^n$ $(n > 1)$, and $\mathcal{G}_{t,q}$, $T_{t,q}$ are the integrals defined by (3.1.1) for $P = P_{t,q}$ and $\Phi = I_{i(t,q)}\varphi_{2n}$.

THEOREM 3.8.1.7. *Let D be a strictly pseudo- convex domain with a boundary $\partial D \in C^\infty$ (or a pseudo -convex domain with a real analytic boundary). Then for any $\overline{\partial}$- closed form $f \in W^{1,2}(\Lambda^{t,q+1}_{|D})$ the series $u = \sum_{\mu=0}^{\infty} \mathcal{G}^{\mu}_{t,q} T_{t,q} \begin{pmatrix} f \\ 0 \end{pmatrix}$ converges in the $W^{1,2}(\Lambda^{t,q}_{|D})$- norm, and*

$$\overline{\partial}_{t,q} u = f, \quad \overline{\partial}^{*}_{t,q-1} u = 0$$

where $\begin{pmatrix} f \\ 0 \end{pmatrix} \in W^{1,2}((\Lambda^{t,q+1}_{|D}, \Lambda^{t,q-1}_{|D}))$.

PROOF. In view of the hypotheses on the domain D, results established in [Kohn] imply that for any $\overline{\partial}$- closed form $f \in W^{1,2}(\Lambda^{t,q+1}_{|D})$ there exists a unique solution $Nf \in W^{2,2}(\Lambda^{t,q+1}_{|D})$ of the $\overline{\partial}$- Neumann problem, and $\overline{\partial}_{t,q}(\overline{\partial}^{*}_{t,q} Nf) = f$ in D. It is clear that $(\overline{\partial}^{*}_{t,q} Nf) \in W^{1,2}(\Lambda^{t,q+1}_{|D})$, and $P_{t,q}(\overline{\partial}^{*}_{t,q} Nf) = \begin{pmatrix} f \\ 0 \end{pmatrix}$. Then Corollary 3.2.4 implies that

$$(\overline{\partial}^{*}_{t,q} Nu) = lim_{\nu \to \infty} \mathcal{G}^{\nu}_{t,q}(\overline{\partial}^{*}_{t,q} Nf) + \sum_{\mu=0}^{\infty} \mathcal{G}^{\mu}_{t,q} T_{t,q} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

and the series f converges in the $W^{1,2}(\Lambda^{t,q}_{|D})$- norm. Therefore one easily obtains $u = (\overline{\partial}^{*}_{t,q} Nf) - lim_{\nu \to \infty} \mathcal{G}^{\nu}_{t,q}(\overline{\partial}^{*}_{t,q} Nf)$ and $P_{t,q} u = P_{t,q}(\overline{\partial}^{*}_{t,q} Nf) = \begin{pmatrix} f \\ 0 \end{pmatrix}$. □

From the proof one can see that the statement holds if for the form $f \in W^{0,2}(\Lambda^{t,q+1}_{|D})$ there exists a form $u \in W^{1,2}(\Lambda^{t,q}_{|D})$ such that $\overline{\partial}_{t,q} u = f$, $\overline{\partial}^{*}_{t,q-1} u = 0$.

Remark 3.8.1.8. Proposition 3.2.5 implies that the series u is the unique solution of the $\overline{\partial}$-equation which belongs to $N^{1,2}(D) \oplus (S^{1,2}_{P_{t,q}}(D))^{\perp}$, where the orthogonal complement is understood in the sense of the special scalar product $H_P(.,.)$ in $S^{1,2}_{\Delta_n I_k}(D)$ (cf. §3.2).

In the case when f is a (0,1)-form Theorem 3.8.1.7 was obtained by A.V. Romanov [Rom2]. In this case the theorem holds for a pseudo -convex domain D with $\partial D \in C^\infty$.

Similar results can be stated for the de Rham complex and for a convex domain D. We consider now interesting case in which D may be non convex.

DEFINITION 3.8.1.9. We say that the Cauchy data $v(q)$ for $u \in W^{1,2}(\Lambda^{q+1}_{|D})$ with respect to the operator d^*_q are equal to zero if, for any $w \in D(\Lambda^{n-q})$, $\int_{\partial D} G_{d^*_q}(w, u) = 0$.

In the following theorem $\mathfrak{H}^{q+1}(\Lambda^{q+1}_{|D}) = \{v \in W^{1,2}(\Lambda^{q+1}_{|D}) : d_{q+1} v = d^*_q v = v_q(v) = 0\}$ are the harmonic spaces and $\mathcal{G}_q$, T_q are the integrals defined by (3.1.1) for $P = P_q$, and $\Phi = I_k \varphi_n$.

THEOREM 3.8.1.10. *Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with a boundary $\partial D \in C^2$ and let $f \in W^{1,2}(\Lambda^{q+1}_{|D})$ be a closed form such that $\int_D (g, f)_x = 0$ for* *any* $g \in \mathfrak{H}^{q+1}(\Lambda^{q+1}_{|D})$. *Then the series* $u = \sum_{\mu=0}^{\infty} \mathcal{G}_q^{\mu} T_q \begin{pmatrix} f \\ 0 \end{pmatrix}$ *converges in the* $W^{1,2}(\Lambda^q_{|D})$*- norm, and*

$$d_q u = f \quad d^*_{q-1} u = 0$$

where $\begin{pmatrix} f \\ 0 \end{pmatrix} \in W^{1,2}((\Lambda^{q+1}_{|D}, \Lambda^{q-1}_{|D}))$.

Proof. This follows from [T5] (p.136) like Theorem 3.8.1.7 from [Kohn]. □

Remark 3.8.1.11. If D is convex then $\mathfrak{H}^q(\Lambda^q_{|D}) = 0$ for $q > 0$.

§3.8.2 The operator $\mathcal{G}$ in the space $S^{s,2}_{I_k\Delta_n}(B_R)$.

We denote by B_R the ball in $\mathbb{R}^n$ with centre at zero and radius $0 < R < \infty$, by σ_n the area of the unit sphere ∂B_1 and by $d\sigma$ the standard volume form on ∂B_R.

Let $S^{s,2}_{I_k\Delta_n}(B_R)$ be the closed subspace of $W^{s,2}(E_{|B_R})$ $(s \geq 0)$ formed by vector functions with harmonic components.

For $s \geq q$, we provide $W^{s,2}(E_{|B_R})$ with the scalar product (cf. 2.8.2)

$$(u,v)_{S^{s,2}_{I_k\Delta_n}(B_R)} = \int_{|y|\leq R} \sum_{j=1}^{k} (u_j, v_j)_{S^{s,2}_{\Delta_n}(B_R)} (u, v \in S^{s,2}_{I_k\Delta_n}(B_R)).$$

Hence, for $s \in \mathbb{Z}_+$, $S^{s,2}_{I_k\Delta_n}(B_R)$ is a Hilbert space with the induced from $W^{s,2}(E_{|B_R})$ Hilbert structure. Moreover, from Proposition 2.8.2.3, the scalar products define a topology, equivalent to the original one in $S^{s,2}_{I_k\Delta_n}(B_R)$ $(s \geq 0)$.

Now, for a vector $u \in S^{0,2}_{I_k\Delta_n}(B_R)$ we denote by $\mathcal{G}u$ its Green's integral in B_R associated with the operator P and the standard fundamental solution $\varphi_n(x)$ of the Laplace operator in $\mathbb{R}^n$:

$$\mathcal{G}u(x) = \frac{1}{R\sigma_n} \lim_{r\to R} \int_{|y|=r} \left(\sum_{i=1} P_i^*(y_i - x_i)\right)\left(\sum_{j=1} P_j y_j\right) \frac{u(y)}{|y-x|^n} d\sigma(y) \ (|x| \neq R).$$

It follows from [ReSz], that the integral $\mathcal{G}$ defines a bounded linear operators $(s \geq 0)$

$$\mathcal{G}_s : S^{s,2}_{I_k\Delta_n}(B_R) \to S^{s,2}_{I_k\Delta_n}(B_R).$$

In this section we are interested in the spectrum of the operators $\mathcal{G}_s$ in $S^{s,2}_{I_k\Delta_n}(B_R)$. We follow in this section the approach of Romanov for the Martinelli-Bochner integral and $s = 1/2$ (see [Rom1]). For this purpose we will use the following lemma.

Lemma 3.8.2.1. *For every homogeneous harmonic polynomial* h_ν *of degree* $\nu \geq 0$ *in* $\mathbb{R}^n$ *we have*

$$\mathcal{G}h_0 = h_0,\ \mathcal{G}h_\nu(x) = h_\nu(x) - \left(\sum_{i=1} P_i^* x_i\right) \frac{(Ph_\nu)(x)}{n+2\nu-2} \ \ (\nu \geq 1). \tag{3.8.2.1}$$

Proof. Since P is a matrix factorization of the Laplace operator in $\mathbb{R}^n$, we have

$$\left(\sum_{i=1} P_i^* x_i\right)\left(\sum_{j=1} P_j x_j\right) = |x|^2 I_k.$$

Hence

$$\mathcal{G}h_\nu(x) = \frac{R}{\sigma_n}\int_{|y|=R}\frac{h_\nu(y)}{|y-x|^n}d\sigma(y)-$$

$$-\left(\sum_{i=1} P_i^* x_i\right)\frac{1}{R\sigma_n}\int_{|y|=R}\left(\sum_{i=1} P_j y_j\right)\frac{h_\nu(y)}{|y-x|^n}d\sigma(y) =$$

$$= \frac{R^2}{R^2-|x|^2}\int_{|y|=R}\mathfrak{P}(x,y)h_\nu(y)d\sigma(y)-$$

$$-\frac{\sum_{i=1} P_i^* x_i}{R^2-|x|^2}\int_{|y|=R}\mathfrak{P}(x,y)(\left(\sum_{i=1} P_j y_j\right)h_\nu(y)d\sigma(y) \tag{3.8.2.2}$$

where $\mathfrak{P}(x,y)$ is the Poisson kernel for the ball B_R:

$$\mathfrak{P}(x,y) = \frac{1}{R\sigma_n}\frac{R^2-|x|^2}{|x-y|^n}.$$

Because the Poisson integral gives the solution of the Dirichlet problem for the ball B_R, we conclude that

$$\frac{R^2}{R^2-|x|^2}\int_{|y|=1}\mathfrak{P}(x,y)h_\nu(y)d\sigma(y) = \frac{R^2\,h_\nu(x)}{R^2-|x|^2}. \tag{3.8.2.3}$$

On the other hand one easily checks that the function (cf. Lemma 2.8.2.1)

$$H(\nu,j,r)(x) = x_j h_\nu^{(r)} - \left(\frac{|x|^2-R^2}{n+2\nu-2}\right)\frac{\partial h_\nu^{(r)}}{\partial x_j}$$

is the harmonic extension of the function $y_j h_\nu^{(r)}(y)$ from ∂B_R to the ball B_R. Therefore

$$\frac{\sum_{i=1} P_i^* x_i}{R^2-|x|^2}\int_{|y|=1}\mathfrak{P}(x,y)\left(\sum_{i=1} P_j y_j\right)h_\nu(y)d\sigma(y) =$$

$$= \frac{\left(\sum_{i=1} P_i^* x_i\right)\left(\sum_{i=1} P_j x_j\right)\,h_\nu(x)}{R^2-|x|^2} + \frac{1}{n+2\nu-2}\left(\sum_{i=1} P_i^* x_i\right)Ph_\nu(x). \tag{3.8.2.4}$$

Now, using (3.8.2.2), (3.8.2.3) and (3.8.2.4) we see that (3.8.2.1) holds. □

We extract from Lemma 3.8.2.1 an information about the spectrum of the operator $\mathcal{G}_0$.

LEMMA 3.8.2.2. *There exists an orthonormal basis* $\{h_\nu^{(i,R)}\}$ *in* $S^{0,2}_{I_k\Delta_n}(B_R)$ ($\nu \geq 0$, $1 \leq i \leq \frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$) *consisting of homogeneous harmonic polynomials with*

$$\mathcal{G}h_\nu^{(i,R)} = \lambda_\nu^{(i,R)}h_\nu^{(i,R)},\ 0 \leq \lambda_\nu^{(i,R)} \leq 1.$$

PROOF. Let us denote by $S_k(\nu)$ the vector space of all the k-vectors of homogoneous harmonic polynomials of degree ν. It is a finite dimensional vector space with $dim S_k(\nu) = \frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ (see [So]). Lemma 3.8.2.1 implies that

$$\mathcal{G}_{|S_k(\nu)} : S_k(\nu) \to S_k(\nu)$$

is a bounded linear operator.

Since $S_k(\nu)$ is finite dimensional, it is a (complex) Hilbert space with the scalar product $(.,.)_{L^2(B_R)}$. On the other hand, due to Lemma 3.8.2.1 and Stokes' formula,

$$(\mathcal{G}h_\nu, g_\nu)_{L^2(B_R)} = (h_\nu, g_\nu)_{L^2(B_R)} - \int_{|y|\le R} (g_\nu)^*(y) \left(\sum_{i=1} P_i^* y_i\right) \frac{(Ph_\nu)(y)}{n+2\nu-2} dy =$$

$$= (h_\nu, g_\nu)_{L^2(B_R)} - \frac{R^2\, (Ph_\nu, Pg_\nu)_{L^2(B_R)}}{(n+2\nu-2)(n+2\nu)}. \tag{3.8.2.5}$$

In particular, this means that the operator $\mathcal{G}_{|S_k(\nu)}$ is selfadjoint. Hence we conclude that *spectr* $\mathcal{G}_{|S_k(\nu)} \subset [-m, m]$, where m is the operator norm of $\mathcal{G}_{|S_k(\nu)}$.

Since the space $S_k(\nu)$ is finite dimensional, there exist $\frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ eigenvectors of the operator $\mathcal{G}_{|S_k(\nu)}$ (corresponding to eigenvalues $\lambda_\nu^{(i,R)}$), which form an orthogonal (with respect to $(.,.)_{L^2(B_R)}$) basis in $S_k(\nu)$.

For $\nu = 0$, $h_0^{(i)} = \sqrt{R^n V_n} 1_i$, $\lambda_0^{(i)} = 1$ $(1 \le i \le k)$, where 1_i is k-vector with components $1_i^j = \delta_{ij}$ and V_n is the volume of the unit ball in $\mathbb{R}^n$.

Let $\nu \ge 1$ ($\nu \ge 2$ if $n = 2$). Because $S_k(\nu)$ is finite dimensional, it is a (complex) Hilbert space with the scalar product (cf. §3.2)

(3.8.2.6)

$$H_P(h_\nu, g_\nu) = \int_{B_R} (Pg_\nu)^*(y)(Ph_\nu)(y)dy + \int_{\mathbb{R}^n \setminus B_R} (PS(g_\nu))^*(y)(PS(h_\nu))(y)dy \ (h_\nu \in S_k(\nu))$$

where $S(h_\nu) = \frac{R^{n+2\nu-2}\, h_\nu(x)}{|x|^{n+2\nu-2}}$ is a harmonic function outside of the ball B_R with zero at infinity and $S(h_\nu) = h_\nu$ on ∂B_R. Then, due to Proposition 3.2.9

$$H_P(\mathcal{G}h_\nu, g_\nu) = \int_{\mathbb{R}^n \setminus B_R} (PS(g_\nu)^*(y) PS(h_\nu))(y)dy, \tag{3.8.2.7}$$

and, in particular, $0 \le H_P(\mathcal{G}h_\nu, h_\nu) \le H_P(h_\nu, h_\nu)$. Hence we conclude that $0 \le \lambda_\nu^{(i,R)} \le 1$ $(1 \le i \le dim S_k(\nu))$.

For the case $n = 2$, $\nu = 1$, we have $\lambda_1^{(i)} = 1 - \mu_1^{(i)}/2$, where $\mu_1^{(i)}$ are eigenvalues of the symmetric block-matrix $Q = \begin{pmatrix} I_k & P_1^* P_2 \\ P_2^* P_1 & I_k \end{pmatrix}$, $\|Q\| = max|q_{mN}| \le 1$, i.e. $0 \le \lambda_1^{(i)} \le 1$.

Because of Lemma 2.8.2.2 it is possible to choose in the space $S^{0,2}_{I_k\Delta_n}(B_R)$ a basis $\{\widetilde{h}_\nu^{(i)}\}$ with $\widetilde{h}_\nu^{(i)} \in S_k(\nu)$ and $1 \le i \le dim S_k(\nu)$. Therefore, because spherical harmonics of different degrees of homogeneity are orthogonal in $S^{0,2}_{I_k\Delta_n}(B_R)$, we can choose an orthogonal basis $\{h_\nu^{(i,R)}\}$ $(\nu \ge 0,\ 1 \le i \le dim S_k(\nu))$ in $S^{0,2}_{I_k\Delta_n}(B_R)$, consisting of the eigenfunctions of the operator $\mathcal{G}$. □

LEMMA 3.8.2.3. *$h_\nu^{(i)} = \sqrt{R^{n+2\nu}} h_\nu^{(i,R)}$ and $\lambda_\nu^{(i)} = \lambda_\nu^{(i,1)} = \lambda_\nu^{(i,R)}$; $\lambda_\nu^{(i)} = 1$ if and only if $Ph_\nu^{(i)} = 0$. Moreover, $h_\nu^{(i)}$ is an orthogonal basis in $S^{s,2}_{I_k\Delta_n}(B_R)$ $(s \geq 0)$.*

PROOF. This is an immediate sequence of the homogeneity of the polynomials, formulae (3.8.2.6), (3.8.2.7) and Proposition 2.8.2.3. □

The following theorem was proved for the Martinelli-Bochner integral in the ball and $s = 1/2$ in Romanov ([Rom1]).

THEOREM 3.8.2.4. *The operator $\mathcal{G}_s : S^{s,2}_{I_k\Delta_n}(B_R) \to S^{s,2}_{I_k\Delta_n}(B_R)$ is a bounded linear selfadjoint operator with spectr $\mathcal{G}_s \subset [0,1]$.*

PROOF. According to Lemmata 3.8.2.2, 3.8.2.3, for a vector $u \in S^{s,2}_{I_k\Delta_n}(B_R)$, we have

$$(\mathcal{G}u, u)_{S^{s,2}_{I_k\Delta_n}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} \lambda_\nu^{(i)} |C_\nu^{(i)}(u)|^2 \|h_\nu^{(i)}\|^2_{S^{s,2}_{I_k\Delta_n}(B_R)} \geq 0,$$

$$\|\mathcal{G}u\|^2_{S^{s,2}_{I_k\Delta_n}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} (\lambda_\nu^{(i)})^2 |C_\nu^{(i)}(u)|^2 \|h_\nu^{(i)}\|^2_{S^{s,2}_{I_k\Delta_n}(B_R)} \leq \|u\|^2_{S^{s,2}_{I_k\Delta_n}(B_R)}$$

because $0 \leq \lambda_\nu^{(i)} \leq 1$ for the eigenvalues of the operators $\mathcal{G}_s$. Here $C_\nu^{(i)}(u)$ are the Fourier coefficients of u with respect to the orthogonal system $\{h_\nu^{(i)}\}$. Therefore $\mathcal{G}_s$ is a selfadjoint operator with *spectr* $\mathcal{G}_s \subset [0,1]$. □

Because of the orthogonality and the completeness of the system $\{h_\nu^{(i)}\}$, $0 \leq \lambda_\nu^{(i)} \leq 1$ are the only eigenvalues of the operators $\mathcal{G}_s$.

Remark 3.8.2.5. In a similar way, using the topological isomorphism between the Hilbert space $S^{s,2}_{I_k\Delta_n}(B_R)$ and the space of harmonic $W^{s,2}(E_{|\mathbb{R}^n \setminus B_R})$-functions $(s \geq 0,\, n \geq 3)$, one can also easily obtain information on the spectrum of Green's integral $\mathcal{G}^{(c)} = \mathcal{G}_{|\mathbb{R}^n \setminus B_R}$. For this it is enough to note that the system $\{\varphi_n(x), \frac{h_\nu^{(i)}(x)}{|x|^{n+2\nu-2}}\}$ $(1 \leq i \leq \frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!},\ \nu \geq 1)$ is the system of eigenvalues of the operator $\mathcal{G}^{(c)}$:

$$\mathcal{G}^{(c)}\varphi_n = 0,\ \mathcal{G}^c \left(\frac{h_\nu^{(i)}}{|x|^{n+2\nu-2}} \right) = \frac{(1-\lambda_\nu^{(i)})\, h_\nu^{(i)}}{|x|^{n+2\nu-2}}.$$

In the case $n = 2$, it is necessary to consider the integral $\mathcal{G}^{(c)}$ in the space of harmonic $W^{s,2}_{loc}(E_{|\mathbb{R}^2 \setminus B_R})$-functions, regular at infinity with respect to the fundamental solution φ_2 (see [Ta], p.45). □

EXAMPLE 3.8.2.6. Let $P = \nabla_n$ $(n \geq 2)$ be the gradient operator in $\mathbb{R}^n$ $(l = n$, $k = 1)$. Then, due to Lemma 3.8.2.1 and Euler formula for homogeneous functions, for every homogeneous harmonic polynomial h_ν we have

$$\mathcal{G}h_\nu = \frac{n+\nu-2}{n+2\nu-2} h_\nu.$$

Arguing as before, we obtain that the multiplicity of the eigenvalue $\frac{1}{2} < \frac{n+\nu-2}{n+2\nu-2} \leq 1$ is $\frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!} < \infty$ and that *spectr* $\mathcal{G}$ consists of the eigenvalues $\frac{n+\nu-2}{n+2\nu-2}$ and the limit point $1/2$, if $n \geq 3$.

In the degenerate case $n = 2$ the spectrum *spectr* $\mathcal{G}$ consists only of two eigenvalues: $1/2$ (eigenvalue of the infinite multiplicity corresponding to $\nu > 0$) and simple eigenvalue 1 (corresponding to $\nu = 0$), i.e $\mathcal{G}$ is not compact. □

EXAMPLE 3.8.2.7. Let $P = 2\overline{\partial}$ be the (doubled) Cauchy-Riemann system in $\mathbb{C}^m$ ($m \geq 2$) written in the complex form with the complex coordinates z_j, $\overline{z}_j$ ($1 \leq j \leq m$). Then $n = 2m$, $l = m$, $k = 1$ and $\mathcal{G}$ is the Martinelli-Bochner integral.

Romanov (see [Rom1]) studied the spectrum of the operator $\mathcal{G}$ in the Hardy spaces $H^2(B_1)$ ($\cong h^{1/2,2}(B_R)$) and $H^2(\mathbb{C}^m \backslash B_1)$ ($\cong h^{1/2,2}(\mathbb{C}^m \backslash B_R)$). He proved that harmonic homogeneous polynomials

$$h_{rt} = \sum_{|\alpha|=r} \sum_{|\beta|=t} c_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}$$

with multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$ and degree of the homogeneity $\nu = r + t$, are the eigenvalues of the operator $\mathcal{G}$:

$$\mathcal{G} h_{rt} = \frac{m + r - 1}{m + r + t - 1} h_{rt},$$

and that we can always choose an orthogonal basis $\{\widetilde{h}_{rt}\}$ ($r \geq 0$, $t \geq 0$) in $H^2(B_1)$ ($\cong h^{1/2,2}(B_R)$) consisting of polynomials of the type h_{rt}.

One easily checks that this implies that all rational numbers of the interval $[0, 1]$ are eigenvalues of infinite multiplicity of the Martinelli -Bochner integral $\mathcal{G}$, and that $\mathit{spectr}\, \mathcal{G}_s = \mathit{spectr}\, \mathcal{G}_s^{(c)} = [0, 1]$. In particular, the operators $\mathcal{G}_s$ and $\mathcal{G}_s^{(c)}$ are not compact.

In the degenerate case $m = 1$ we have $n = 2$, $l = 1$, $k = 1$ and $\mathcal{G}$ is the Cauchy integral. The spectrum $\mathit{spectr}\, \mathcal{G}$ consists only of two eigenvalues (both are of infinite multiplicity): 1 (eigenvalue corresponding to z^{ν}, $\nu \geq 0$), and 0 (eigenvalue corresponding to $\overline{z}^{\nu}$, $\nu > 0$), i.e $\mathcal{G}$ is not compact.

Kytmanov [Ky] proved that the (doubled) singular Martinelli-Bochner integral

$$\mathcal{G}_b(z) = 2\mathcal{G}(z) \ (z \in \partial B_1)$$

defines a selfadjoint bounded linear operator in the Lebesgue space $L^2(\partial B_1)$, with

$$\mathcal{G}_b h_{rt} = \frac{m + r - t - 1}{m + r + t - 1} h_{rt}.$$

Hence all the rational numbers of the interval $(-1, 1]$ are eigenvalues of infinite multiplicity of the singular Martinelli-Bochner integral $\mathcal{G}_b$, and $\mathit{spectr}\, \mathcal{G}_b = [-1, 1]$. $\square$

Now we will use the information about the spectrum of the operators $\mathcal{G}_s : S^{s,2}_{I_k \Delta_n}(B_R) \to S^{s,2}_{I_k \Delta_n}(B_R)$ ($s \geq 0$) to obtain Theorem on Iterations (cf. §3.1, §3.2).

Due to Theorem 1.4.4, for $u \in S^{s,2}_{I_k \Delta_n}(B_R)$ ($s \geq 0$) there exists weak boundary value $Pu_{|\partial B_R}$ belonging to the Sobolev space $W^{s-3/2,2}(E_{|\partial B_R})$. Let us denote by $\tau P u$ the single layer potential:

$$(\tau P u)(x) = \frac{1}{R} \lim_{r \to R} \int_{|y|=r} \left(\sum_{j=1} P_j^* y_j \right) \varphi_n(x - y)(Pu)(y) d\sigma(y).$$

By Stokes' formula we have

$$(\mathcal{G}u)(x) + (\tau P u)(x) = \begin{cases} u(x), x \in B_R, \\ 0, x \in X \backslash \overline{B}_R \end{cases} \tag{3.8.2.8}$$

for every $u \in S^{s,2}_{I_k\Delta_n}(B_R)$. Therefore the integral τP defines linear bounded operators $\tau_s P$ from $S^{s,2}_{I_k\Delta_n}(B_R)$ to $S^{s,2}_{I_k\Delta_n}(B_R)$.

In particular, it is possible to consider iterations $\mathcal{G}^\nu = \mathcal{G} \circ \mathcal{G} \circ \dots \circ \mathcal{G}$, $(\tau P)^\nu = (\tau P) \circ (\tau P) \circ (\dots) \circ (\tau P)$ ($\nu \geq 1$ times) of the integrals $\mathcal{G}$ and τP in these spaces.

As before, $S^{s,2}_P(B_R)$ is the closed subspace of $S^{s,2}_{I_k\Delta_n}(B_R)$ consisting of solutions of the system $Pu = 0$ in B_R. Then $\Pi(S^{s,2}_P(B_R))$ stand for the orthogonal projections from $S^{s,2}_{I_k\Delta_n}(B_R)$ to $S^{s,2}_P(B_R)$.

Since $S^{s,2}_{I_k\Delta_n}(B_R)$ is topologically isomorphic to $S^{s,2}_{I_k\Delta_n}(\mathbb{R}^n \backslash B_R)$ ($n \geq 3$), we associate $u \in S^{s,2}_{I_k\Delta_n}(B_R)$ a (unique) vector function $S(u) \in S^{s,2}_{I_k\Delta_n}(\mathbb{R}^n \backslash B_R)$ with $u = S(u)$ on ∂B_R.

In the case where $n = 2$ we associate $u \in S^{s,2}_{I_k\Delta_n}(B_R)$ a (unique) vector function $S(u)$, harmonic in $\mathbb{R}^2 \backslash B_R$, regular at infinity with respect to φ_n (see [Ta4]) and such that $u = S(u)$ on ∂B_R.

Then we can consider $S^{s,2}_P(\mathbb{R}^n \backslash B_R) = \{u \in S^{s,2}_{I_k\Delta_n}(B_R) : \ PS(u) = 0 \ in \ \mathbb{R}^n \backslash B_R\}$ as a closed subspace of $S^{s,2}_{I_k\Delta_n}(B_R)$ and $\Pi(S^{s,2}_P(\mathbb{R}^n \backslash B_R))$ stands for the orthogonal projection from $S^{s,2}_{I_k\Delta_n}(B_R)$ to $S^{s,2}_P(\mathbb{R}^n \backslash B_R)$.

For the Martinelli-Bochner intefgral and $s \in \mathbb{Z}_+$ this fact was mentioned in [Ky].

Theorem 3.8.2.8 (on Iterations). *Let $s \geq 0$. Then*

$$\lim_{\nu \to \infty} {\mathcal{G}_s}^\nu = \Pi(S^{s,2}_P(B_R)), \quad \lim_{\nu \to \infty} (\tau_s P)^\nu = \Pi(S^{s,2}_P(\mathbb{R}^n \backslash B_R))$$

in the strong operator topology in $S^{s,2}_{I_k\Delta_n}(B_R)$.

Proof. It follows immediately from Theorem 3.1.2 and Theorem 3.8.2.4. □

Let us consider, for $s \geq m \geq s - 1$ the linear closed densely defined operator

$$P_{s,m} : W^{s,2}(E_{|B_R}) \to W^{m,2}(F_{|B_R}).$$

And let now $dom\, P_{s,m} = \{u \in W^{s,2}(E_{|B_R}) : (Pu) \in W^{m,2}(F_{|B_R})\}$. It is easy to see that $dom\, P_{s,m}$ is a Hilbert space with the scalar product

$$(u, v)_{W^{s,2}(E_{|B_R})} + (Pu, Pv)_{W^{m,2}(F_{|B_R})} \ (u, v \in dom\, P_{s,m}).$$

Let T be the following integral:

$$Tf(x) = \frac{1}{\sigma_n} \int_{B_R} \left(\sum_{j=1}^{n} P^*_j (y_j - x_j) \right) \frac{f(y)}{|y - x|^n} dy \ (f \in L^2(F_{|B_R})).$$

The integral T defines bounded linear operators $T_m : W^{m,2}(F_{|B_R}) \to W^{m+1,2}(E_{|B_R})$ (see Lemma 3.2.7). Hence the composition TP defines a bounded linear operator $T_m P_{s,m} : dom\, P_{s,m} \to W^{m+1,2}(E_{|B_R})$.

Now we can define an extension $\widetilde{\mathcal{G}}_s$ of the operator $\mathcal{G}_s : S^{s,2}_{I_k\Delta_n}(B_R) \to S^{s,2}_{I_k\Delta_n}(B_R)$ for $u \in dom\, P_{s,m}$. Indeed, if $u \in dom\, P_{s,m}$ then there exists a sequence $u_N \in C^s(E_{|\overline{B}_R})$ such that

$$\lim_{N \to \infty} \|u - u_N\|_{|W^{s,2}(E_{|B_R})} + \|Pu - Pu_N\|_{|W^{m,2}(F_{|B_R})} = 0.$$

Then, for $u \in dom\, P_{s,m}$, we set

$$\widetilde{\mathcal{G}}_s u = \lim_{N\to\infty} \frac{1}{R\sigma_n} \int_{|y|=R} \left(\sum_{i=1} P_i^*(y_i - x_i)\right) \left(\sum_{j=1} P_j y_j\right) \frac{u_N(y)}{|y-x|^n} d\sigma(y).$$

Stokes' formula and the continuity of the operator T_m imply that, for $u \in dom\, P_s$,

$$\widetilde{\mathcal{G}}_s u = \lim_{N\to\infty} (u_N - TPu_N) = u - (T_m P_{s,m})u. \tag{3.8.2.9}$$

Hence the operator $\widetilde{\mathcal{G}}_s$ is well defined and does not depend on the choice of the sequence u_N.

LEMMA 3.8.2.9. *$\widetilde{\mathcal{G}}u = \mathcal{G}u$ for $u \in S^{s,2}_{I_k\Delta_n}(B_R) \cap dom\, P_{s,m}$.*

PROOF. It follows from Stokes' formula, that $TPu = \tau Pu$ for $u \in S^{s,2}_{I_k\Delta_n}(B_R) \cap dom\, P_{s,m}$. Hence (3.8.2.8) and (3.8.2.9) imply that $\mathcal{G}u = \widetilde{\mathcal{G}}u$. $\square$

Then Lemma 3.8.2.9 and Theorem 3.8.2.8 imply the folowing result (similar to Corollary 3.1.3).

COROLLARY 3.8.2.10. *For every $u \in dom\, P_{s,m}$ we have*

$$u = \lim_{\nu\to\infty} \widetilde{\mathcal{G}}^\nu u + \sum_{\mu=0}^{\infty} \widetilde{\mathcal{G}}^\mu TPu = \lim_{\nu\to\infty} (TP)^\nu u + \sum_{\mu=0}^{\infty} (TP)^\mu (\widetilde{\mathcal{G}}u),$$

where the limits and the series converge in the $W^{s,2}(E_{|B_R})$-norm.

Now we obtain a formula for solutions of $Pu = f$ in B_R whenever they exists in $dom\, P_{s,m}$ (cf. §3.3).

COROLLARY 3.8.2.11. *Let $f \in W^{m,2}(F_{|B_R})$ such that $Pv = f$ in B_R with $v \in dom\, P_{s,m}$ $(s-1 \le m \le s)$. Then the series*

$$u = \sum_{\mu=0}^{\infty} G^\mu Tf = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1, Ph_\nu^{(i)}\neq 0}^{dim S_k(\nu)} \frac{(n+2\nu-2)(n+2\nu)\, C_\nu^{(i)}(\widetilde{\mathcal{G}}Tf)}{\|Ph_\nu^{(i)}\|^2_{L^2(B_1)}} h_\nu^{(i)}, \tag{3.8.2.10}$$

where

$$C_\nu^{(i)}(\widetilde{\mathcal{G}}Tf) =$$

$$= \frac{1}{R\sigma_n} \frac{1}{(n+2\nu)(n+2\nu-2)} \int_{|y|=R} \left(\sum_{m=1}^{n} P_m^* \frac{\partial}{\partial y_m} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}\right) \left(\sum_{j=1}^{n} P_j y_j\right) (Tf)(y) d\sigma(y)$$

are the Fourier coefficients of the vector $\widetilde{\mathcal{G}}Tf$ with respect to the orthogonal basis $\{h_\nu^{(i)}\}$ in $S^{0,2}_{I_k\Delta_n}(B_R)$, converges in the $W^{s,2}(E_{|B_R})$-norm and $Pu = f$ in B_R.

PROOF. The formula

$$u = \sum_{\mu=0}^{\infty} G^\mu Tf = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1, \lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} \frac{C_\nu^{(i)}(\widetilde{\mathcal{G}}Tf)}{1-\lambda_\nu^{(i)}} h_\nu^{(i)}$$

follows from Corollary 3.8.2.10 and the fact that

$$\Pi(S^{s,2}_P(B_R))\widetilde{\mathcal{G}}Tf = \lim_{\nu\to\infty} \widetilde{\mathcal{G}}^\nu(\widetilde{\mathcal{G}}Tf) = \lim_{\nu\to\infty} \widetilde{\mathcal{G}}^\nu(\widetilde{\mathcal{G}}Tf - \widetilde{\mathcal{G}}^2 Tf) = 0. \tag{3.8.2.11}$$

In order to finish the proof we note that, according to (3.8.2.5),

$$\frac{\|Ph^{(i)}_\nu\|^2_{L^2(B_1)}}{1-\lambda^{(i)}_\nu} = (n+2\nu-2)(n+2\nu). \quad \square$$

We emphasize that the coefficients $C^{(i)}_\nu$ in (3.8.2.11) do not depend on s and m. In the next section we discuss in detail the existence of $W^{s,2}(E_{|B_R})$-solutions of the equation $Pu = f$ and obtain a formula for its solutions with data in $W^{m,2}(F_{|B_R})$ $(m \geq 0)$.

§**3.8.3 On the solvability of the system $Pu = f$ in B_R in a ball.**

As in §§3.3, 3.4, in this section we obtain a criterion for the existence of $W^{s,2}(E_{|B_R})$-solutions of the system (cf. §3.3, §3.4).

$$Pu = f \ in \ B_R$$

and a formula for its $W^{m+1,2}_{loc}(E_{|B_R})$-solutions with the datum $f \in W^{m,2}(F_{|B_R})$ $(m \geq 0)$.

Because P is a system of partial differential operators with constant coefficients and injective principal symbol, it can be included into an elliptic Hilbert compatibility complex

$$0 \to C^\infty(E) \xrightarrow{P} C^\infty(F) \xrightarrow{P^1} C^\infty(G) \xrightarrow{P^2} \dots. \tag{3.8.3.1}$$

with $P^\circ = P$. This means that P^1 is a differential operator with constant coefficients of order $p_1 \geq 1$,

$$P^{i+1}\circ P^i = 0$$

and that

$$\mathbb{C}^k \xrightarrow{\sigma(P)(\zeta)} \mathbb{C}^l \xrightarrow{\sigma_{p_1}(P^1)(\zeta)} \mathbb{C}^N$$

is an exact sequence for every $\zeta \in \mathbb{R}^n\setminus\{0\}$.

One easily sees that the condition $P^1 f = 0$ is a necessary one for the solvability of the equation $Pu = f$.

Let us denote by $S^{m,2}_{P^1,P^*}(B_R)$ the following closed subspace of $W^{m,2}(F_{|B_R})$:

$$S^{m,2}_{P^1,P^*}(B_R) = \{g \in W^{m,2}(F_{|B_R}): \ P^1 g = 0, \ P^* g = 0 \ in B_R\}.$$

Lemma 3.8.3.1. *The system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ is an orthogonal basis in the Hilbert space $S^{m,2}_{P^1,P^*}(B_R)$. Moreover there exist constants $\widetilde{C}_1(m,n)$, $\widetilde{C}_2(m,n) > 0$ such that*

$$\widetilde{C}_1(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(F_{|B_R})} \leq \nu^{2m}\|Ph^{(i)}_\nu\|^2_{L^2(F_{|B_R})} \leq \widetilde{C}_2(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(F_{B_R})}.$$

Proof. It follows from the Stokes' formula and Lemmata 3.8.2.1, 3.8.2.2 that

$$(Ph^{(i)}_\nu, Ph^{(j)}_\mu)_{L^2(B_R)} = \int_{|y|=R} (h^{(j)}_\mu)^*(y) \left(\sum_{m=1}^n P^*_m x_m \right) Ph^{(i)}_\nu(y) d\sigma(y) =$$

$$= (n+\nu-1)(1-\lambda^{(i)}_\nu)(h^{(i)}_\nu, h^{(j)}_\mu)_{L^2(F_{|B_R})}.$$

Therefore the system $\{Ph^{(i)}_\nu\}$ is orthogonal $L^2(F_{|B_R})$.

On the other hand, using the Stokes' formula and the homogeneity of the polynomials, one easily has, for $\nu \geq m$,

$$\sum_{|\alpha|=m} (D^\alpha Ph^{(i)}_\nu, D^\alpha Ph^{(j)}_\mu)_{L^2(B_R)} = (\nu - m) \sum_{|\beta|=m-1} (D^\beta Ph^{(i)}_\nu, D^\beta Ph^{(j)}_\mu)_{L^2(\partial B_R)} =$$

$$\frac{(\nu-m)(n+\nu+\mu-2m+1)}{R^{n+\mu+\nu-2m+1}} \sum_{|\beta|=m-1} (D^\beta Ph^{(i)}_\nu, D^\beta Ph^{(j)}_\mu)_{L^2(F_{|B_R})}.$$

This formula implies immediately the orthogonality in $W^{1,2}(F_{|B_R})$ and arguing by induction we obtain the orthogonality in $W^{m,2}(F_{|B_R})$ ($m \in \mathbb{Z}_+$).

The estimates follows immediately from the calculations above.

Since the compatibility complex (3.8.3.1) is elliptic, for every $g \in S^{m,2}_{P^1,P^*}(B_R)$, there exists $v \in W^{m+1,2}_{loc}(B_R)$, satisfying $Pv = g$ in B_R (see, for example, [AnNa]). In particular, for every $0 < r < R$, $v \in S^{s,2}_{I_k \Delta_n}(B_r)$,

$$v = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} c^{(i)}_\nu (v,r) h^{(i)}_\nu,$$

where the series converges in the $W^{s,2}(E_{|B_r})$-norm (and hence, due to Stiltjes-Vitaly Theorem, uniformly together with all the derivatives on compact subsets of B_r),

$$g = Pv = \sum_{\nu=1}^{\infty} \sum_{i=1, \lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} c^{(i)}_\nu (v,r) Ph^{(i)}_\nu.$$

Because of Lemma 3.8.2.3, the coefficients $c^{(i)}_\nu(v,r)$ do not depend on r. Moreover, due to the orthogonality the system $\{Ph^{(i)}_\nu\}_{\lambda^{(i)}_\nu \neq 1}$, $c^{(i)}_\nu(v,r)$ depend only on g and do not depend on v. Hence the statement of the lemma holds. $\square$

Now, for $m \geq 0$ ($m \notin \mathbb{Z}_+$) we provide the space $S^{m,2}_{P^1,P^*}(B_R)$ with the Hermitian form

$$(u,v)_{S^{m,2}_{P^1,P^*}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} K^{(i)}_\nu(f) \overline{K^{(i)}_\nu(g)} \nu^{2m} \ (f, g \in S^{m,2}_{P^1,P^*}(B_R)),$$

where $K^{(i)}_\nu(f)$ are the Fourier coefficients of the vector-function f with respect to the orthonormal basis $\{Ph^{(i)}_\nu\}$ in $S^{0,2}_{P^1,P^*}(B_R)$.

The following proposition follows from Lemma 3.8.3.1 and Proposition 2.8.2.3.

Proposition 3.8.3.2. *For every $m \geq 0$, the Hermitian form $(.,.)_{S^{s,2}_{I_k\Delta_n}(B_R)}$ is a scalar product in $S^{m,2}_{I_k\Delta_n}(B_R)$ defining the topology, equivalent to the original one. Moreover, the system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ is an orthogonal basis in the Hilbert space $S^{m,2}_{P^1,P^*}(B_R)$ and there exist constants $\widetilde{C}_1(m,n)$, $\widetilde{C}_2(m,n) > 0$ such that*

$$\widetilde{C}_1(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(F_{|B_R})} \leq \nu^{2m}\|Ph^{(i)}_\nu\|^2_{L^2(F_{|B_R})} \leq \widetilde{C}_2(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(F_{|B_R})}.$$

In the following corollary $K^{(i)}_\nu(f-PTf)$, are the Fourier coefficients of the vector $f-PTf$ with respect to the orthogonal system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu\neq 0}$ in $L^2(F_{|B_R})$:

$$K^{(i)}_\nu(f-PTf) = \frac{((f-PTf),\, Ph^{(i)}_\nu)_{L^2(F_{|B_R})}}{\|Ph^{(i)}_\nu\|^2_{L^2(F_{|B_R})}}.$$

Corollary 3.8.3.3. *For every $f \in W^{m,2}(F_{|B_R})$, with $P^1 f = 0$ in B_R the vector-function*

$$u = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1,\lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} K^{(i)}_\nu(f-PTf)h^{(i)}_\nu,$$

where the series converges in $W^{s,2}(E_{|B_r})$-norm for every $0 < r < R$, satisfies $Pu = f$ in B_R.

Proof. Since $T : W^{m,2}(F_{|B_R}) \to W^{m+1,2}(E_{|B_R})$, using Stokes' formula one easily checks that

$$P^*(f-PTf) = 0,\ P^1(f-PTf) = 0 \ in\ B_R.$$

Moreover, Proposition 3.8.3.2 implies that, for every $m \geq 0$,

$$\frac{((f-PTf),\, Ph^{(i)}_\nu)_{L^2(F_{|B_R})}}{\|Ph^{(i)}_\nu\|^2_{L^2(F_{|B_R})}} = \frac{((f-PTf),\, Ph^{(i)}_\nu)_{W^{m,2}(F_{|B_R})}}{\|Ph^{(i)}_\nu\|^2_{W^{m,2}(F_{|B_R})}} \ (Ph^{(i)}_\nu \neq 0).$$

Now, arguing as in the proof of Lemma 3.8.3.1, we obtain that the statement of the corollary holds. □

Theorem 3.8.3.4. *Let $m \geq 0$ and $0 \leq s \leq m+1$. Then the following conditions are equivalent:*

(1) *for every $f \in W^{m,2}(F_{|B_R})$, with $P^1 f = 0$ in B_R there exists $u \in W^{s,2}(E_{|B_R})$, satisfying $Pu = f$ in B_R;*

(2)
$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s} \left| \frac{C^{(i)}_\nu(\widetilde{\mathcal{G}}Tf)}{1-\lambda^{(i)}_\nu} \right|^2 < \infty$$

for every $f \in W^{m,2}(F_{|B_R})$, with $P^1 f = 0$ in B_R.

(3)
$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s}\left|K_\nu^{(i)}(f-PTf)\right|^2 < \infty$$

for every $f \in W^{m,2}(F_{|B_R})$, *with* $P^1 f = 0$ *in* B_R*;*

(4) *there exists a constant* $C_1 > 0$ *such that*

$$\max_{\lambda_\nu^{(i)}\neq 1} \frac{1}{1-\lambda_\nu^{(i)}} \leq C_1 \nu^{2-2s+2m} \ for\ every\ \nu \geq 1,\ 1 \leq i \leq dim S_k(\nu);$$

(5) *there exists a constant* $C_2 > 0$ *such that*

$$\min_{Ph_\nu^{(i)}\neq 0} \|Ph_\nu^{(i)}\|^2_{L^2(F_{|B_1})} \geq C_2 \nu^{2s-2m} \ for\ every\ \nu \geq 1,\ 1 \leq i \leq dim S_k(\nu).$$

Proof. If for every $f \in W^{m,2}(F_{|B_R})$, with $P^1 f = 0$ in B_R, there exists a section $u \in W^{s,2}(E_{|B_R})$, satisfying $Pu = f$ in B_R then, according to Corollary 3.8.2.11, the series
$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} \frac{C_\nu^{(i)}(\tilde{\mathcal{G}}Tf)}{1-\lambda_\nu^{(i)}} h_\nu^{(i)}$$

converges in the $W^{s,2}(E_{|B_R})$-norm for every $f \in W^{m,2}(F_{|B_R})$, with $P^1 f = 0$ in B_R. Therefore, (see Proposition 2.8.2.3 and Lemma 3.8.2.3)

$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s}\left|\frac{C_\nu^{(i)}(\tilde{\mathcal{G}}Tf)}{1-\lambda_\nu^{(i)}}\right|^2 < \infty$$

for every $f \in W^{m,2}(E_{|B_R})$, with $P^1 f = 0$ in B_R, i.e (1) implies (2).

Because of the orthogonality and the homogeneity of the system $\{Ph_\nu^{(i)}\}$,

$$\frac{C_\nu^{(i)}(\tilde{\mathcal{G}}Tf)}{1-\lambda_\nu^{(i)}} = K_\nu^{(i)}(f-PTf),\ (Ph_\nu^{(i)} \neq 0\ in\ B_R).$$

Hence (2) and (3) are equivalent.

Let (3) holds. We fix $f \in W^{m,2}(E_{|B_R})$, with $P^1 f = 0$ in B_R. Then, according to Corollary 3.8.3.3, the vector-function

$$u = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} K_\nu^{(i)}(f-PTf) h_\nu^{(i)},$$

satisfies $Pu = f$ in B_R and, with a constant $C_2(s,n) > 0$ depending only on s and n (see Proposition 2.8.2.3),

$$\|u\|^2_{W^{s,2}(E_{|B_R})} \leq \|Tf\|^2_{W^{s,2}(E_{|B_R})} +$$

$$+\frac{1}{C_1(s,n)}\sum_{\nu=1}^{\infty}\sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s}\left|K_\nu^{(i)}(f-PTf)\right|^2<\infty,$$

i.e. (3) implies (1).

Condition (5) implies that, with a constant $\widetilde{C}(m,n)>0$ depending only on m and n (see Lemma 3.8.2.3 and Proposition 2.8.2.3),

$$\sum_{\nu=1}^{\infty}\sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s}\left|K_\nu^{(i)}(f-PTf)\right|^2\leq$$

$$\leq\frac{1}{C_2}\sum_{\nu=1}^{\infty}\sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)}\left|K_\nu^{(i)}(f-PTf)\right|^2 R^{2\nu}\nu^{2m}\|Ph_\nu^{(i)}\|^2_{L^2(B_1)}\leq$$

$$\leq\frac{\widetilde{C}_2(m,n)}{C_2}\sum_{\nu=1}^{\infty}\sum_{i=1,\lambda_\nu^{(i)}\neq 1}^{dim S_k(\nu)}\left|K_\nu^{(i)}(f-PTf)\right|^2\|Ph_\nu^{(i)}\|^2_{W^{m,2}(F_{|B_R})}<\infty,$$

i.e. (5) implies (3).

Further, Proposition 3.8.3.2 and Corollary 3.8.3.3 imply that the image $\overline{Im\,P}$ of the operator

$$P:S^{s,2}_{I_k\Delta_n}(E_{|B_R})\to S^{m,2}_{P^1,P^*}(F_{|B_R})$$

is closed. Then (1) yields that there exists a constant $C_0>0$ such that

$$\|u\|^2_{W^{s,2}(E_{|B_R})}\leq C_0\|Pu\|^2_{W^{m,2}(F_{|B_R})}$$

for every $u\in\left(S^{s,2}_P(B_R)\right)^{\perp}$, with $(Pu)\in W^{m,2}(F_{|B_R})$, where $\left(S^{s,2}_P(B_R)\right)^{\perp}$ is the orthogonal complement of $S^{s,2}_P(B_R)$ in $S^{s,2}_{I_k\Delta_n}(E_{|B_R})$ (cf. [Hö]). In particular,

$$\frac{R^{2\nu}\nu^{2s}}{C_2(s,n)}\leq\|h_\nu^{(i)}\|^2_{W^{s,2}(E_{|B_R})}\leq C_0\|Ph_\nu^{(i)}\|^2_{W^{m,2}(F_{|B_R})}\leq\frac{C_0\,R^{2\nu}\,\nu^{2m}}{\widetilde{C}_1(m,n)}\|Ph_\nu^{(i)}\|^2_{L^2(F_{|B_1})}$$

for every $h_\nu^{(i)}$ with $\lambda_\nu^{(i)}\neq 1$. Therefore (1) implies (5).

Finally, (3.8.2.5) implies that (4) and (5) are equivalent. □

Corollary 3.8.3.5. *One can find a finite number $a\geq -1$ (depending on the operator P) that, for every $f\in W^{a+s,2}(F_{|B_R})$ $(s\geq 0,\ a+s\geq 0)$ satisfying $P^1f=0$ in B_R, there exists a $W^{s,2}(E_{|B_R})$-solution u to $Pu=f$ in B_R.*

Proof. It follows, for example, from Lemma 2.8.2.1 and Proposition 3.8.2.5 that the system $\{h_\nu^{(i)}\}$ is a basis in the space $C^\infty(E_{|\overline{B}_R})\cap S_{I_k\Delta_n}(B_R)$ of harmonic vector-functions in B_R belonging to $C^\infty(E_{|\overline{B}_R})$. Then, for every $u\in C^\infty_{I_k\Delta_n}(\overline{B}_R)$, the series

$$u=\sum_{\nu=0}^{\infty}\sum_{i=1}^{dim S_k(\nu)}C_\nu^{(i)}(u)h_\nu^{(i)}$$

converges in $C^\infty(E_{|\overline{B}_R})$, and the series

$$u_1 = \sum_{\nu=1}^{\infty} \sum_{i=1, Ph_\nu^{(i)} \neq 0}^{dim S_k(\nu)} C_\nu^{(i)}(u) h_\nu^{(i)}$$

converges in $W^{s,2}(E_{|B_R})$ for every $s \geq 0$. According to Sobolev Embedding Theorems, $u_1 \in C^q(E_{|\overline{B}_R})$ for every $q \geq 0$, i.e. $u_1 \in C^\infty(E_{|\overline{B}_R})$. Hence one easily conclude that the series u_1 converges in $C^\infty(E_{|\overline{B}_R)}$.

It is known (see, for example, [AnNa]) that, for every $g \in C^\infty(F_{|\overline{B}_R})$ satisfying $P^1 g = 0$ in B_R, there exists a vector $v \in C^\infty(E_{|\overline{B}_R})$ with $Pv = g$ in B_R. Therefore the operator

$$P_\infty : \left(C_P^\infty(\overline{B}_R)\right)^\perp \to \{g \in C^\infty(F_{|\overline{B}_R}) : \ P^1 g = 0, \ P^* g = 0 \ in \ B_R\}$$

(where $\left(C_P^\infty(\overline{B}_R)\right)^\perp$ stands for the closure of the linear span of the system $\{h_\nu^{(i)}\}_{Ph_\nu^{(i)} \neq 0}$ in $C^\infty(E_{|\overline{B}_R}))$ is injective, surjective and continuous. The Open Mapping Theorem for the Frechet spaces implies the inverse operator P_∞^{-1} of P_∞ is continuos too.

Now, using Theorem 3.8.3.4, one easily concludes that there exists such a finite number $a \geq -1$ (depending on the operator P) that, for every $f \in W^{a+s,2}(F_{|B_R})$ ($s \geq 0$, $a+s \geq 0$) satisfying the integrability conditions, there exists a $W^{s,2}(E_{|B_R})$-solution u to $Pu = f$ in B_R, unless the operator P_∞^{-1} is not continuos. $\square$

Corollary 3.8.3.5 can be proved using Ehrenpreis Fundamental Principle (see [Bj]).

Example 3.8.3.6. Let $n_1 \geq 1$, $n_2 \geq 1$, Q be $l_1 \times k$ matrix factorization of the Laplace operator in $\mathbb{R}_x^{n_1}$ and q be $l_2 \times 1$ matrix factorization of the Laplace operator in $\mathbb{R}_y^{n_2}$. Then the operator

$$P = \begin{pmatrix} Q_x \\ \frac{1}{k} q_y \otimes I_k \end{pmatrix}$$

is a matrix factorization of the Laplace operator in $\mathbb{R}^{n_1+n_2}$.

We assume that either the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ or the dimension of the vector space $S_q(\mathbb{R}^{n_2})$ is not finite. Then, Theorem 3.8.3.4 implies that, for every $m \geq 0$ and $s > m + 1/2$, the image $Im(P_{s,m})$ of the operator

$$P_{s,m} : W^{s,2}(E_{|B_R}) \to W^{m,2}(F_{|B_R})$$

is not closed (cf Example 3.6.4 and [Ke] for the Cauchy-Riemann system).

Indeed, let the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ be not finite. We fix an eigenfunction $\widetilde{h}_1(y)$ of Green's operator $\mathcal{G}_q$ corresponding to a ball in $\mathbb{R}^{n_2}$ and to an eigenvalue $\widetilde{\lambda}_1 \neq 1$. Because the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ is not finite, for any number $N > 0$ there exists a number $\nu \geq N$ such that $Q\widetilde{h}_\nu(x) = 0$ in B_R, and therefore there exists a harmonic homogeneous polynomial $h_{\nu+1} = \widetilde{h}_1(y) h_\nu(x)$ in $\mathbb{R}^{n_1+n_2}$, with

$$\mathcal{G}_P h_{\nu+1} = \lambda_{\nu+1} h_{\nu+1}, \quad \frac{1}{1 - \lambda_{\nu+1}} = \frac{n_1 + n_2 + 2\nu}{(n_1 + n_2)(1 - \widetilde{\lambda}_1)}.$$

Due to Theorem 3.8.3.4, $Im(P_{s,m})$ is not closed for every $s > m + 1/2$.

The proof for the case, where the dimension of the vector space $S_q(\mathbb{R}^{n_2})$ is not finite, is similar. $\square$

EXAMPLE 3.8.3.7. Let $n = 3$, $\mathbb{R}^3 = \mathbb{C}^1_z \times \mathbb{R}_x$, $l = 2$, $k = 1$,

$$P = \begin{pmatrix} 2\frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

According to Example 3.8.3.6 we can not guarantee the existence of $W^{s,2}(B_R)$-solutions of $Pu = f$ for all data in $W^{m,2}(F_{|B_R})$ satisfying the compatibility conditions if $s > m + 1/2$. However we can do it for $s = m + 1/2$.

Indeed, one easily checks that harmonic homogeneous polynomials of the type

$$h_\nu = \sum_{r+t+N=\nu} x^N h_{r,t}$$

(where h_{rt} are the polynomials from Example 3.8.2.9) are dense in $h^{s,2}(B_R)$. Moreover,

$$\mathcal{G} h_\nu = \frac{1 + N + 2r}{1 + 2\nu} h_\nu.$$

Now Theorem 3.8.3.4 implies that for all $W^{m,2}(F_{|B_R})$-data satisfying the compatibility conditions there exist $W^{m+1/2,2}(B_R)$-solutions of the equation $Pu = f$. $\square$

As in §3.5, we can easily apply the spectral decomposition of the operator G_s and Theorem 3.8.3.4 to the following P-Neumann Problem (cf. also [Ky], §§17–19).

PROBLEM 3.8.3.8. *Let $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$ be a given vector, $m \geq 0$ and $0 \leq N \leq m + 1$. It is requared to find $u \in W^{N,2}(E_{|B_R})$ such that*

$$\left\{ \begin{array}{l} \Delta_n I_k u = 0 \ \mathit{in}\ B_R \\ \left(\sum_{i=1}^n P_i^* x_i\right) P u = \psi \ \mathit{on}\ B_R. \end{array} \right. \tag{3.8.3.2}$$

It follows from the Stokes' formula that the condition

$$\int_{|y|=R} (h_\nu^{(i)})^*(y)\psi(y) d\sigma(y) = 0 \ \mathit{for\ all}\ h_\nu^{(i)} \ \mathit{with}\ P h_\nu^{(i)} = 0 \tag{3.8.3.3}$$

is necessary, for Problem 3.8.3.8 to be solvable. Because the dimension of $S_P(\mathbb{R}^n)$ can be infinite, in general, Problem 3.8.3.8 is not an elliptic boundary value problem.

In the following corollary $\tilde{\tau}\psi$ stands for the integral

$$(\tilde{\tau}\psi)(x) = \frac{1}{R} \int_{|y|=R} \varphi_n(x - y)\psi(y) d\sigma(y)$$

and $C_\nu^{(i)}(\tilde{\tau}\psi)$ stands for the Fourier coefficients of the vector $\tilde{\tau}\psi$ with respect to the orthogonal basis $\{h_\nu^{(i)}\}$ in $S^{0,2}_{I_k\Delta_n}(B_R)$ (because $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$, using 2.3.2.5 in [ReSz], we conclude that $\tilde{\tau}\psi \in S^{m+1,2}_{I_k\Delta_n}(B_R)$).

COROLLARY 3.8.3.9. *If Problem 3.8.3.8 is solvable for every $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$, satisfying (3.8.3.3), then one of the conditions in Theorem 3.8.3.4 hold with $s = N$ and*

$$u = \sum_{\nu=1}^{\infty} \sum_{i=1, P h_\nu^{(i)} \neq 0}^{dim S_k(\nu)} \frac{(n + 2\nu - 2)(n + 2\nu)\, C_\nu^{(i)}(\tilde{\tau}\psi)}{R^{n+2\nu}\, \| P h_\nu^{(i)} \|^2_{L^2(B_1)}} h_\nu^{(i)}, \tag{3.8.3.4}$$

is its unique solution in $\left(S_P^{N,2}(B_R)\right)^{\perp}$. *Back, if conditions of Theorem 3.8.3.4 hold with* $s = (m+N+1)/2$, *Problem 3.8.3.8 is solvable for every* $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$, *satisfying (3.8.3.3).*

PROOF. Since $\tau P u = \widetilde{\tau\psi}$ for a solution u of Problem 3.8.3.8, using Corollary 3.8.2.9 one easily concludes that formula (3.8.3.4) holds.

Let Problem 3.8.3.8 be solvable for every $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$, satisfying (3.8.3.3). Then, it is easy to see that (3.8.3.3) holds for $\psi = (\sum_{i=1}^n P_i^* x_i)\, f \in W^{m-1/2,2}(E_{|\partial B_R})$ with $f \in S^{m,2}_{P^1,P^*}(\partial B_R)$. Denoting by $u \in S^{N,2}_{I_k\Delta_n}(B_R)$ a solution of Problem 3.8.3.8 for such a vector ψ, we obtain that $\tau P u - \widetilde{\tau}\,(\sum_{i=1}^n P_i^* x_i)\, f = 0$. Because of Lemma 3.8.2.1, Propositions 3.8.2.5 and 3.8.3.2, $f = Pu$, i.e. the conditions of Theorem 3.8.3.4 holds with $s = N$.

Back, because $\psi \in W^{m-1/2,2}(E_{|\partial B_R})$, there exists a function $v \in S^{m,2}_{I_k\Delta_n}(B_R)$ such that $v_{|\partial B_R} = \psi$. Condition (3.8.3.3) implies $= v \in \left(S_P^{m,2}(B_R)\right)^{\perp}$. Hence we can decompose v with respect to the orthogonal basis $\{h_\nu^{(i)}\}_{Ph_\nu^{(i)} \neq 0}$ in this space. Denoting by $C_\nu^{(i)}(v)$ the corresponding Fourier coefficients, we set

$$\widetilde{u} = \sum_{\nu=1}^{\infty} \sum_{i=1, Ph_\nu^{(i)} \neq 0}^{dim S_k(\nu)} \frac{(n+2\nu)\, C_\nu^{(i)}(v)}{\|Ph_\nu^{(i)}\|^2_{L^2(B_1)}} h_\nu^{(i)}.$$

One easily calculates that $\widetilde{u} = u$ is a solution of Problem 3.8.3.8, if condition (5) of Theorem 3.8.3.4 holds with $s \geq (m+N+1)/2$. □

COROLLARY 3.8.3.10. *For every* $\psi \in C^\infty(E_{|\partial B_R})$, *satisfying (3.8.3.3), there exists* $u \in C^\infty_{I_k\Delta_n}(\overline{B}_R)$ *satisfying (3.8.3.2).*

In the case where P is the gradient operator in $\mathbb{R}^n$, Problem 3.8.3.8 is the Neumann Problem and (3.8.3.4) is a classical formula for its solutions (cf., for example, [Vl], p. 426–428). For the Cauchy-Riemann system see [Ky], p. 181). æ
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