ON THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION

A.A. Shlapunov

Introduction

Let D be a bounded domain in $\mathbb{R}^n$ and S be a closed smooth hypersurface dividing it into 2 connected components: D^+ and $D^- = D$, and oriented as the boundary of D^-.

Problem 1. *Under what conditions on functions $f_0 \in C^1(S)$ and $f_1 \in C^0(S)$ is there a function $f \in C^1(D^- \cup S)$, which is harmonic in D^- and such that the restrictions on S of f and its normal derivative $\frac{\partial f}{\partial n}$ are equal to f_0 and f_1 correspondingly ?*

It is well known that Problem 1 is unstable. Nevertheless, contrary to Hadamard's famous stetement (see [1], p.38) it is often met with in applications. There is a sizable literature on the subject (see, e.g. [2]-[6]). Tarkhanov [7] has published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In this paper we will describe a simpler (necessary and sufficient) conditions (which is more easy to verify) for Problem 1 to be solvable.

In §1 we state the criterion. In §2, under the assumption that f is a function in $L^2(D^-)$, we formulate the criterion on the language of special bases which have the property of double orthogonality (see [8]). In §3, as an example, we construct such a basis. In §4 we study the case where $D = B(x^0, R)$ is a ball in $\mathbb{R}^n$. Finally in §5, we present Carleman's formula for the determination of a harmonic function in D^-, given its data on S.

§1. Criterion for solvability of Problem 1

We denote by σ_n the area of the unit sphere in $\mathbb{R}^n$ and by $g(y)$ the standard (bilateral) fundamental solution of the Laplace operator in $\mathbb{R}^n$:

$$g(y) = \begin{cases} \frac{1}{(2-n)\sigma_n |y|^{n-2}}, & n > 2, \\ \frac{1}{2\pi} ln|y|, & n = 2. \end{cases}$$

Assume that the functions f_0, f_1 are summable on S. Then the corresponding Green's integral is well defined:

$$\mathcal{F}(x) = \int_S \left(f_0(y) \frac{\partial g(x-y)}{\partial n_y} - f_1(y) g(x-y) \right) ds(y) \ (x \in D \backslash S).$$

It is clear that $\mathcal{F}$ is harmonic everywhere outside of S; let $\mathcal{F}^{\pm} = \mathcal{F}_{|D^{\pm}}$.

1991 *Mathematics Subject Classification.* 35J55,.

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 33, No. 3, pp. 205- 215, May -June, 1992. Original article submitted may 21, 1991. 0037-4466/92/3303-0534$ 12.50 1992 Plenum Publishing Corporation

Lemma 1.1. *Let $S \in C^2$, $f_0 \in C^1$ and $f_1 \in C^0$ be summable functions on S. Then the function $\mathcal{F}^+$ continuously extends to $D^+ \cup S$ together with its first derivatives if and only if the function $\mathcal{F}^-$ continuously extends to $D^- \cup S$ together with its first derivatives.*

Proof. We will use the fact that there exist a smooth function $\hat{f}$ given in a neighbourhood of S in D such that $\hat{f}_{|S} = f_0$, $\frac{\partial \hat{f}}{\partial n}_{|S} = f_1$ (see [9], Lemma 29.5).

If $x^0 \in S$, $\nu(x^0)$ is the unit normal vector to S at the point x^0 and $|\alpha| \leq 1$ then (see [9], Lemma 29.5)

$$\lim_{\varepsilon \to 0} (\partial^\alpha \mathcal{F})(x^0 - \varepsilon \nu(x^0)) - \partial^\alpha \mathcal{F}(x^0 + \varepsilon \nu(x^0))) = \partial^\alpha \hat{f}(x^0), \tag{1.1}$$

where the limit is uniform on compact subsets in S.

Let, for instance, $\mathcal{F}^-$ continuously extends to $D^- \cup S$ together with its first derivatives. We fix a multi-index $|\alpha| \leq 1$. Then

$$\lim_{\varepsilon \to 0} \partial^\alpha \mathcal{F}(x^0 + \varepsilon \nu(x^0)) = \partial^\alpha \mathcal{F}(x^0) - \partial^\alpha \hat{f}(x^0).$$

Let us define $\mathcal{F}^+$ in the following way:

$$\partial^\alpha \mathcal{F}^+(x) = \begin{cases} \partial^\alpha \mathcal{F}^+(x), \ x \in D^+, \\ \partial^\alpha \mathcal{F}^-(x) - \partial^\alpha \hat{f}(x), \ x \in S. \end{cases}$$

Let us show that $\partial^\alpha \mathcal{F}^+$ is continuous in $D^- \cup S$. We fix a point $x^0 \in S$ and $E > 0$. Because $\partial^\alpha \mathcal{F}^+$ is continuous on S, there is $\delta_0 > 0$ such that, for $x^1 \in S$ with $|x^1 - x^0| < \delta_0$, we have

$$|\partial^\alpha \mathcal{F}^+(x^1) - \partial^\alpha \mathcal{F}^+(x^0)| < E/2.$$

Decreasing δ_0 (if it is necessary) we can consider $K = \overline{B(x^0, \delta_0)} \cap S$ as a compact subset of S.

Since the hypersurface $S \in C^2$, we can choose $0 < \delta < \delta_0$ in such a way that every point $x \in D^+ \cap B(x^0, \delta)$ is represented in the form $x = x^1 + \varepsilon \nu(x^1)$ where $x^1 \in S$ and $\varepsilon = dist(x, S)$. Then $\varepsilon < \delta$ and $|x^0 - x^1| \leq |x^0 - x| + |x - x^1|$, i.e. $x^1 \in K$.

Using the fact that the limit in (1.1) is uniform on compact subsets of S and decreasing δ (if it is necessary) we obtain that, for $x^1 \in K$, $0 < \varepsilon < \delta$ the following inequality holds:

$$|\partial^\alpha \mathcal{F}^+(x^1 + \varepsilon \nu(x^1)) - \partial^\alpha \mathcal{F}^+(x^1)| < E/2.$$

Let now $x \in D^+ \cap B(x^0, \delta)$. Then, for some $x^1 \in K$ and $0 < \varepsilon < \delta$ we have $x = x^1 + \varepsilon \nu(x^1)$. Hence

$$|\partial^\alpha \mathcal{F}^+(x^0) - \partial^\alpha \mathcal{F}^+(x)| \leq |\partial^\alpha \mathcal{F}^+(x^0) - \partial^\alpha \mathcal{F}^+(x^1)| +$$

$$+ |\partial^\alpha \mathcal{F})^+(x^1 + \varepsilon \nu(x^1)) - \partial^\alpha \mathcal{F}^+(x^1)| < E.$$

Therefore $\mathcal{F}^+$ continuously extends to $D^+ \cup S$ together with its first derivatives, if $\mathcal{F}^-$ continuously extends to $D^- \cup S$ together with its first derivatives. The proof is complete. □

Theorem 1.2. *Let $S \in C^2$, $f_0 \in C^1$ and $f_1 \in C^0$ be summable functions on S. Then, for Problem 1 to be solvable, it is necessary and sufficient that the integral $\mathcal{F}^+$ harmonically extends from D^+ to the domain D.*

Proof. Necessity. Suppose that there exists a function f that solves Problem 1. define in D the function

$$\Phi(x) = \begin{cases} \mathcal{F}^+(x), \ x \in D^+ \\ \mathcal{F}^- - f(x), \ x \in D^-. \end{cases} \tag{1.2}$$

For any subdomain $S_1 \subset S$ ther eis some domain $D_1 \Subset D$ in D^- with a piecewise-smooth boundary such that $S_1 \subset \partial D_1$. Clearly, $f \in C^1(\overline{D}_1)$ is harmonic in D_1 and so, by Green's formula,

$$f(x) = \int_{\partial D_1} \left(f(y) \frac{\partial g(x-y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} g(x-y) \right) ds(y) \ (x \in D_1).$$

Hence we have, in D^-,

$$\Phi(x) = \mathcal{F}^-(x) - f(x) = \int_{S \setminus S_1} \left(f_o(y) \frac{\partial g(x-y)}{\partial n_y} - f_1(y) g(x-y) \right) ds(y) +$$

$$+ \int_{\partial D_1 \setminus S_1} \left(f(y) \frac{\partial g(x-y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} g(x-y) \right) ds(y) \ (x \in D_1). \tag{1.3}$$

The terms in the right hand side of (1.3) are harmonic functions in a neighbourhood of S_1, and therefore, since S_1 is arbitrary, $\mathcal{F}^-$ extends smoothly to $D^- \cap S$.

Further, it follows from Lemma 1.1 that $\mathcal{F}^+$ also extends smoothly to $D^+ \cup S$. Therefore, the restriction $\Phi^\pm$ to $D^\pm$ of ϕ extends smoothly to $D^\pm \cup S$. In addition, by (1.1), if $x^0 \in S$, then

$$\lim_{\varepsilon \to 0} \Phi^-(x^0 - \varepsilon\nu(x^0)) - \Phi^+(x^0 + \varepsilon\nu(x^0)) = 0,$$

$$\lim_{\varepsilon \to 0} \frac{\partial \Phi^-}{\partial n}(x^0 - \varepsilon\nu(x^0)) - \frac{\partial \Phi^+}{\partial n}(x^0 + \varepsilon\nu(x^0)) = 0.$$

Thus, we conclude that Φ can be extended smoothly to the whole domain D by defining $\Phi = \mathcal{F}^- - f$ on S.

By Morera's theorem for harmonic functions, it follows that a function Φ that is smooth in D and harmonic in D^- and in D^+ is also harmonic in D. By (1.2) Φ is the desired harmonic extension of $\mathcal{F}^+$ to D.

Sufficiency. Let $\mathcal{F}^+$ be extendable to a harmonic function in D, call it Φ. Then, by Lemma 1.1, $\mathcal{F}^-$ extends smoothly to $D^- \cup S$. Define $f(x) = \mathcal{F}^- - \Phi(x)$ $(x \in D^-)$. Using formula (1 .1) as in the proof of the necessity, we see that the restriction to S of f and its normal derivative $\frac{\partial f}{\partial n}$ equal f_0 and f_1, respectively. □

Example 1.3. Let S be a piece of the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n$. Then, if $f_0 = 0$, the function $\mathcal{F})$ is even with respect to $x_n \neq 0$, and, if $f_1 = 0$, it is odd. Therefore, if one of the functions f_j $(0 \leq j \leq 1)$ is zero, the integrals $\mathcal{G}(\oplus f_j)^{\pm}$ extend harmonically across S simultaneously. Because their difference on S is equal to f_0, and the difference of their normal derivatives is equal to f_1, Theorem 1.2 implies the known Hadamard's statement (see [17]. p. 31). Namely, if one of the functions f_j $(0 \leq j \leq 1)$ is zero, Problem 1 is solvable only if another function is real analytic.

Remark 1.4. The fact that $D \subset \mathbb{R}^n$ is a bounded domain is essential in this section only for $n = 2$, because of the construction of the fundamental solution of the Laplace operator.

§2. Solvability of Problem 1 in L^2 in a domain in terms of bases with double orthogonality

In this section we will assumed that the surface S can be extended smoothly to a neighbourhood of $\overline{D}$, and that f_0, $f_1 \in L^2(S)$ are diven functions on S.

Let $G \Subset D^+$ be a domain with piecewise-smooth boundary such that the complement of G has no compact connected components in D. Let $h^2(G)$ denote the space of harmonic $L^2(G)$-functions, with induced topology.

We consider a system of functions $\{b_\nu\}$ in $h^2(G)$, possessing special properties: $\{b_\nu\}$ is an orthonormal basis in $h^2(D)$ and an orthogonal basis in $h^2(G)$. It was shown in [8] that under the conditions above such bases with double orthogonality exists, and a method for their construction was established.

We will use the system $\{b_\nu\}$ to solve the following problem.

Problem $1'$. *Under what conditions on functions $f_0 \in C^1(S)$ and $f_1 \in C^0(S)$ is there a function $f \in C^1(D^- \cup S) \cap h^2(D^-)$, such that the restrictions on S of f and its normal derivative $\frac{\partial f}{\partial n}$ are equal to f_0 and f_1, respectively ?*

Clearly, the restriction to G of $\mathcal{F}^+$ belongs to $h^2(G)$. Let c_ν denote the Fourier coefficients of $\mathcal{F}^+$ with respect to the orthogonal system $\{b_\nu\}$. Since G and S are disjoint, these coefficients can be written in the following form:

$$c_\nu = \left(\int_G \mathcal{F}^+(x) \overline{b_\nu(x)} dx \right\backslash \left(\int_G |b_\nu(x)|^2 dx \right) =$$

$$= \int_S \left(f_0(y) \frac{\partial}{\partial n_y} \frac{\int_G g(x-y) \overline{b_\nu(x)} dx}{\int_G |b_\nu(x)|^2 dx} - f_1(y) \frac{\int_G g(x-y) \overline{b_\nu(x)} dx}{\int_G |b_\nu(x)|^2} \right) ds(y)$$

Theorem 2.1. *Let $S \in C^2$. Then Problem $1'$ is solvable if and only if the series $\sum_{\nu=1}^{\infty} |c_n u|^2$ is convergent.*

Proof. Necessity. Suppose that there exist a solution of Problem $1'$. By the conditions we have imposed on S, $L^2(S) \subset L^1(S)$. Therefore it follows from Theorem 1.2 that the function $\mathcal{F}^+$ extends to a harmonic function in D, say Φ. In addition, it was proved in Theorem 1.2, that the extension Φ has the form

$$\Phi(x) = \begin{cases} \mathcal{F}^+(x), & x \in D^+ \\ \mathcal{F}^- - f(x), & x \in D^- \cup S. \end{cases}$$

Now, due to the conditions imposed on S, we may assume that D^- is contained in a domain D_1 with smooth boundary such that $S \subset D_1$. Since the extension of f_0 by zero to $\partial D_1 \backslash S$ belongs to $L^2(D_1)$, it follows from results of [10] that $\mathcal{F}^- \in L^2(D_1)$; in particular $\mathcal{F}^- \in L^2(D^-)$. Arguing similarly we obtain that $\mathcal{F}^+ \in L^2(D^+)$. Thus $\Phi \in h^2(D)$ and the expansion $\mathcal{F}^+(x) = \sum_{\nu=1}^{\infty} c_n u b_\nu(x)$ still converges in the norm of $L^2(D)$. By Bessel's inequality, $\sum_{\nu=1}^{\infty} |c_n u|^2 \leq \|\Phi\|_{L^2(D)} < \infty$.

Sufficiency. Suppose that the series $\sum_{\nu=1}^{\infty} |c_n u|^2$ is convergent. Then, by the Riesz-Fischer theorem, there is a function $\Phi \in h^2(D)$ such that $\Phi(x) = \sum_{\nu=1}^{\infty} c_n u b_\nu(x)$. Clearly, Φ is a harmonic extension of $\mathcal{F}^+$. By Theorem 1.2, the function $f(x) = \mathcal{F}^-(x) - \Phi(x)$ $(x \in D^-)$ is a solution of Problem 1. It remains to observe that, by arguments above, $\mathcal{F}^- \in L^2(D)$, and hence $f \in h^2(D^-)$. This completes the proof. □

§3. Example of basis with double orthogonality

Let $O = B_R$ be the ball with centre at zero and radius $0 < R < \infty$, and S be a closed smooth hypersurface dividing it into 2 connected components (D^+ and D^-) in such away that $0 \in D^+$, and oriented as the boundary of D^-. In this case we can construct a basis with double orthogonality in the subspace of $L^2(B_R)$, which consists of harmonic functions in a rather explicit form.

Let $\{h_\nu^{(i)}\}$ be a set of homogeneous harmonic polynomials which form a complete orthonormal system in $L^2(\partial B_1)$ where ν is the degree of homogeneity, and i is an index labelling the polynomials of degree ν belonging to the basis. The size of the index set for i as a function of ν is known, namely, $1 \leq i \leq J(\nu)$ where $J(\nu) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ for $n > 2$ and $\nu = 0$. If $n = 2$ then, obviously, $J(0) = 1$, $J(\nu) = 2$ for $\nu \geq 1$. Using the system $\{h_\nu^{(i)}\}$ we will construct the basis with double orthogonality.

In the folowing lemma $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{b_\nu\}$.

Lemma 3.1. *Let $h = h(\alpha)$ be a continuous map of a topological space $\mathcal{A}$ to $\mathcal{H}$. Then, for any element $h(\alpha)$, the Fourier series converges uniformly with respect to α on compact subsets of $\mathcal{A}$.*

Proof. Let $(.,.)$ be the scalar product and $\|h\| = (h,h)^{1/2}$ be a norm in $\mathcal{H}$ $(h \in \mathcal{H})$.

We fix arbitrary $\alpha \in \mathcal{A}$ and denote by $c_\nu(\alpha)$ the Fourier coefficients of the vector $h(\alpha)$ with respect to the system $\{b_\nu\}$: $c_\nu(\alpha) = (h(\alpha), b_\nu)$. Then for any $\varepsilon > 0$ there is $N > 0$, $N = N(\varepsilon, \alpha)$, such that for every $m \geq N$ the following inequality holds:

$$\|h(\alpha) - \sum_{\nu=1}^{m} c_\nu(\alpha) b_\nu\| = \left(\|h(\alpha)\|^2 - \sum_{\nu=1}^{m} |c_\nu(\alpha)|^2 \right)^{1/2} \leq \varepsilon. \tag{3.1}$$

Since the map h and the scalar product $(.,.)$ are continuous, there is a neighbourhood $\mathcal{V}_N(\alpha)$ of the point α in which estimate (3.1) still holds for $m = N$. However, if m increases, the right hand side of (3.1) can only decrese. Therefore inequality (3.1) holds in the neighbourhood $\mathcal{V}_N(\alpha)$ for all $m \geq N$.

Now, for any compact $K \subset \mathcal{A}$, we can choose $N_1 = N_1(K)$ such that estimate (3.1) holds for all $\alpha \in K$ because we can cover the compact by a finite number of neighbourhoods of the type $\mathcal{V}_N(\alpha)$. The proof is complete. □

Lemma 3.2. *The fundamental solution of the Laplace operator can be expanded as follows:*

$$g(x-y) = g(y) - \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h^{(i)}_\nu(x)}{n+2\nu-2}\frac{\overline{h^{(i)}_\nu(y)}}{|y|^{n+2\nu-2}}. \tag{3.2}$$

where the series converges together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |y| > |x|\}$.

Proof. Because of the homogeneity of the polynomial $h^{(i)}_\nu$, Euler formula implies that

$$\sum_{m=1}^{n} \frac{\partial h^{(i)}_\nu}{\partial x_m} x_m = \nu h^{(i)}_\nu, \; \sum_{m=1}^{n} \frac{\partial^2 h^{(i)}_\nu}{\partial x_m \partial x_j} x_m = (\nu-1)\frac{\partial h^{(i)}_\nu}{\partial x_j} x_j. \tag{3.3}$$

We denote by $Y^{(i)}_\nu$ the restriction of the polynomyal $h^{(i)}_\nu$ to ∂B_1. Then $\{Y^{(i)}_\nu\}$ is a basis in $L^2(\partial B_1)$ consisting of spherical functions.

Let $x \in B_1$ be fixed. We represent $\varphi_n(x-y)$ by the Fourier series in $L^2(\partial B_1)$. Namely,

$$\varphi_n(x-y) = \sum_{\nu,i} c^{(i)}_\nu(x)\overline{Y^{(i)}_\nu},$$

where $c^{(i)}_\nu(x)$ are the Fourier coefficients of $\varphi_n(x-y)$ with respect to the system $\{Y^{(i)}_\nu\}$.

Let us consider first the case where $n > 2$. Then

$$c^{(i)}_\nu(x) = \frac{1}{(2-n)\sigma_n}\int_{\partial B_1} |x-y|^{2-n} Y^{(i)}_\nu(y) d\sigma(y),$$

where $d\sigma$ is the volume form on the sphere ∂B_1. We rewrite the coefficients in the following way:

$$c^{(i)}_\nu(x) = \frac{1}{(2-n)}\int_{\partial B_1} \mathfrak{P}(x,y)\frac{1-2<x,y>+|x|^2}{1-|x|^2} Y^{(i)}_\nu(y) d\sigma(y). \tag{3.4}$$

Here $<x,y> = \sum_{m=1}^n x_m y_m$ and

$$\mathfrak{P}(x,y) = \frac{1}{\sigma_n}\frac{1-|x|^2}{|x-y|^n}$$

is the Poisson kernel for the unit ball in $\mathbb{R}^n$.

It is not difficult to see that the function

$$\mathcal{F} = x_m h^{(i)}_\nu(x) - \frac{1}{n+2\nu-2}\frac{\partial h^{(i)}_\nu}{\partial x_m}(|x|^2-1) \tag{3.5}$$

is the harmonic extension into the ball B_1 of the function $y_m Y(i)_\nu$ given on ∂B_1. Really, using (3.3) and harmonicity of $h(i)_\nu$ we have:

$$\Delta_n \mathcal{F} = 2\frac{\partial h^{(i)}_\nu}{\partial x_m}(x) - \frac{1}{n+2\nu-2}\frac{\partial h^{(i)}_\nu}{\partial x_m}(x)\Delta_n(|x|^2-1) +$$

$$+\frac{2}{n+2\nu-2}\sum_{j=1}^{n}\frac{\partial^2 h_\nu^{(i)}}{\partial x_m \partial x_j}(x)\frac{\partial}{\partial x_j}(|x|^2-1)=$$

$$=2\frac{\partial h_\nu^{(i)}}{\partial x_m}(x)-\frac{2}{n+2\nu-2}\left(n\frac{\partial h_\nu^{(i)}}{\partial x_m}(x)+2\sum_{j=1}^{n}\frac{\partial^2 h_\nu^{(i)}}{\partial x_m\partial x_j}(x)x_j\right)=0.$$

Using the Poisson formula and equalities (3.3), (3.4) and (3.5) we obtain

$$c_\nu^{(i)}(x)=\frac{1}{(2-n)}\frac{1+|x|^2}{1-|x|^2}\int_{\partial B_1}\mathfrak{P}(x,y)Y_\nu^{(i)}(y)d\sigma(y)-$$

$$-\frac{2}{(2-n)}\sum_{m=1}^{n}\frac{x_m}{1-|x|^2}\int_{\partial B_1}\mathfrak{P}(x,y)y_m Y_\nu^{(i)}(y)d\sigma(y)=-\frac{h_\nu^{(i)}(x)}{n+2\nu-2}.$$

Therefore

$$\varphi_n(x-y)=-\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)}\frac{h_\nu^{(i)}(x)\overline{Y_\nu^{(i)}(y)}}{n+2\nu-2},$$

and Lemma 3.1 implies that this series converges in the norm of the space $L^2(\partial B_1)$, uniformly with respect to x on compact subsets of the ball B_1.

The harmonic extension with respect to y leads us to the equality

$$|y|^{2-n}\varphi_n(x-\frac{y}{|y|})=-\sum_{\nu=0}^{\infty}\sum_{i=1}^{J(\nu)}\frac{h_\nu^{(i)}(x)\overline{h_\nu^{(i)}(y)}}{n+2\nu-2},$$

where the series converges absolutely and uniformly with respect to x and y inside the ball B_1.

Applying to this equality the Kelvin transformation with respect to y we see that

$$\varphi_n(x-y)=-\sum_{\nu=0}^{\infty}\sum_{i=0}^{J(\nu)}\frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}. \tag{3.6}$$

It is clear that series (3.6) converges uniformly with respect to x (inside the ball B_1) and y (outside $\overline{B}_1$). Let us show that it is converges uniformly on the set of the following type

$$\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\frac{|y|}{|x|}\geq\delta_1,\ and\ |y|\geq\delta_0\}$$

where $\delta_1>1$, $\delta_0>0$. We choose $\gamma>1$ such that $\gamma^2<\delta_1$. Then

$$\sum_{i=0}^{J(\nu)}\frac{h_\nu^{(i)}(x)}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}=$$

$$=\left(\frac{\gamma}{|y|}\right)^{n-2}\sum_{i=0}^{J(\nu)}\frac{h_\nu^{(i)}(\frac{x}{\gamma|x|})}{(n+2\nu-2)}\frac{\overline{h_\nu^{(i)}(\frac{y}{\gamma|y|})}\left(\frac{\gamma^2|x|}{|y|}\right)^{\nu}}{|y|^{n+2\nu-2}}.$$

By the choice of γ we have:

$$\left|\frac{x}{\gamma|x|}\right| = \frac{1}{\gamma} < 1, \ \left|\frac{\gamma y}{|y|}\right| = \gamma > 1, \ \frac{\gamma^2|x|}{|y|} \leq \frac{\gamma^2}{\delta_1} < 1.$$

Using the criterion of Abel for the uniform convergence of series, we see that series (3.6) uniformly converges on subsets of the type above.

If $\nu = 0$ then $J(0) = 1$ and $h_0^{(1)} = const$. Because the system $\{h_\nu^{(i)}\}$ is orthonormal we conclude that $|h_0^{(1)}|^2 = \frac{1}{\sigma_n}$. Therefore

$$\varphi_n(x-y) = \frac{1}{(2-n)\sigma_n |y|^{n-2}} - \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{(n+2\nu-2)} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.$$

In the case $n = 2$, we have

$$c_\nu^{(i)}(x) = \frac{1}{2\pi}\int_{\partial B_1} Y_\nu^{(i)}(y) ln|x-y| d\sigma(y).$$

However, from the discussion above, we see that, for $\nu \geq 1$ and $m = 1, 2$,

$$\frac{\partial c_\nu^{(i)}}{\partial x_m}(x) = \frac{1}{2\pi}\int_{\partial B_1} \frac{x_m - y_m}{|y-x|^2} Y_\nu^{(i)}(y) d\sigma(y) = \frac{-1}{2\nu}\frac{\partial h_\nu^{(i)}}{\partial x_m}(x).$$

Moreover, because $\nu \geq 1$, $c_\nu^{(i)}(0) = h_\nu^{(i)}(0) = 0$. Hence

$$c_\nu^{(i)}(x) = -\frac{h_\nu^{(i)}(x)}{2\nu} \ (\nu \geq 1)$$

If $\nu = 0$ then

$$\frac{\partial c_1^{(1)}}{\partial x_m}(x) = \frac{h_0^{(1)}}{2\pi}\int_{\partial B_1} \frac{x_m - y_m}{|y-x|^2} Y_\nu^{(i)}(y) d\sigma(y) =$$

$$= \frac{h_0^{(1)}}{2\nu(1-|x|^2)}\left(x_m \int_{\partial B_1} \mathfrak{P}(x,y) d\sigma(y) - \int_{\partial B_1} y_m \mathfrak{P}(x,y) d\sigma(y)\right) = 0 \ (m = 1, 2).$$

Arguing as before we obtain:

$$\frac{1}{2\pi} ln|x-y| = \frac{1}{2\pi} ln|y| - -\sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{(n+2\nu-2)} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.$$

Finally, because the summands in decomposition (3.2) are harmonic with respect to x and y in $\mathcal{K}$, the Stiltjes-Vitali theorem yields that series (3.2) converges uniformly together with all the derivatives on compact subsets of the cone $\mathcal{K}$. □

A similar decomposition of the fundamental $g(x-y)$ in $\mathbb{C}^n$ maybe found for in [12]).

Remark 3.3. The statement that series (3.2) converges uniformly together with all the derivatives on compact subsets of the cone $\mathcal{K}$ may also be deduced from the following estimate of a homogeneous harmonic polinomial $h_\nu^{(i)}$ of degree ν on the sphere ∂B (see [11]):

$$\max_{|y|=1} |h_\nu^{(i)}| \leq const(n) \nu^{n/2-1} \|h_\nu^{(i)}\|_{L^2(\partial D_1)}.$$

Lemma 3.4. *For any* $0 < R < \infty$

$$(h_\nu^{(i)}, h_\mu^{(j)})_{L^2(B_R)} = \begin{cases} R^{n+2\nu}/(n+2\nu) \ \nu = \mu, \ \textit{and } i = j, \\ 0, \ \nu \neq \mu \textit{ or } j \neq i. \end{cases}$$

Proof.

$$\int_{B_R} h_\nu^{(i)}(x)\overline{h_\mu^{(j)}}(x)dx = \int_0^R dr \int_{|x|=r} h_\nu^{(i)}(x)\overline{h_\mu^{(j)}}(x)d\sigma(x) =$$

$$= \int_0^R r^{\nu}+\mu+n-1dr \int_{|x|=1} h_\nu^{(i)}(x)\overline{h_\mu^{(j)}}(x)d\sigma(x) = \begin{cases} R^{n+2\nu}/(n+2\nu) \ \nu = \mu, \ \textit{and } i = j, \\ 0, \ \nu \neq \mu \textit{ or } j \neq i. \quad \square \end{cases}$$

Lemma 3.5. *For any ball B centered at zero, the system $\{h_\nu^{(i)}\}$ is complete in the space $h^2(B)$.*

Proof. Let B an arbitrari ball centered at zero and $f \in h^2(B)$. It is known that $f \in h^2(B)$ can be approximated in the norm of the space $L^2(B)$ by functions f_N ($N = 1, 2, ...$), which are harmonic in a neighbourhood of the ball $\overline{B}$ (see, for example, [79], ch. 4). Because, for every ($N = 1, 2, ...$), the function f_N is harmonic in a neighbourhood of a (larger than B) ball $\hat{B}$, it can be represented in the ball $\hat{B}$ by Green's formula. Replacing the fundamental solution $g(x-y)$ in this Green's formula by decomposition (3.2), we obtain a sequence $\{f_{NM}\}$ of finite linear combinations of polynomials $h_\nu^{(i)}$ which converges to f_N in the norm of $L^2(B)$. Taking the diagonal sequence $\{f_{NN}\}$ we obtain the desired approximation of f in the norm of $L^2(B)$. The proof is complete. $\square$

Theorem 3.6. *For any $0 < r < \infty$ the system $\{Q_\nu^{(i)}\} = \{\sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h_\nu^{(i)}\}$ is an orthonormal basis in $h^2(B_r)$ and an orthogonal basis in $h^2(B)$ where B is an arbitrary ball with centre at zero.*

Proof. Follows immediately from Lemmata 3.4, 3.5. $\square$

§4. Solvability criterion for a ball

Let $D = B_R$ be a ball in $\mathbb{R}^n$ and S be a closed hypersurface in D, dividing it into 2 connected components D^+ and D^-, with the origin in D^+. We fix $0 < r < dist(0, S)$ and set $\Omega = B_r$ so that $\Omega \Subset O$. In order to obtain the Fourier coefficients for the section $\mathcal{F}$ with respect to this basis in $h^2(B_r)$ it is sufficient to know the Fourier coefficients for the fundamental solution $\varphi_n(x-y)$ (see Lemma 2.8.5.).

Our principal results will be formulated in the language of the coefficients

$$k_\nu^{(i)} = \begin{cases} \frac{-1}{n+2\nu-2} \int_S \left(f_0(y) \frac{\partial}{\partial n} \left(\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \right) - f_1(y) \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \right) ds(y) \ (\nu = 1, 2, ...), \\ \int_S \left(f_0(y) \frac{\partial \varphi_n(y)}{\partial n} - f_1(y) \varphi_n(y) \right) ds(y), \nu = 0. \end{cases}$$

Theorem 4.1. *Let f_0, $f_1 \in L^1(S)$. Then for Problem 2.8.1 to be solvable, it is necessary and sufficient that*

$$\limsup_{\nu\to\infty} \max_i \sqrt[\nu]{|k_\nu^{(i)}(y)|} \leq \frac{1}{R} \tag{4.1}$$

Proof. Necessity. Let Problem 1 be solvable. Then Theorem 1.2 implies that the function $\mathcal{F}^+$ on the domain D^+ harmonically extends to a function $\mathcal{F} \in S_{\Delta_n}(B_R)$.

We fix $0 < r < R$. It is clear that the components of the solution $\mathcal{F}$ belong to the space $S^2_{\Delta_n}(B_r)$. Therefore, from Theorem 3.6, they are represented by their Fourier series with respect to the system $\{\sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h_\nu^{(i)}\}$

$$\mathcal{F}(x) = \sum_{i,\nu} c_\nu^{(i)}(r) \sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h_\nu^{(i)}(x) \quad (x \in B_r). \tag{4.2}$$

Bessel's inequality implies that the series $\sum_{i,\nu} |c_\nu^{(i)}(r)|^2$ converges. On the other hand, in the ball Ω, from Lemma 3.2, we obtain the decomposition

$$\mathcal{F}(x) = \sum_{i,\nu} k_\nu^{(i} h_\nu^{(i}(x) \quad (x \in \Omega). \tag{4.3}$$

Comparing (4.2) and (4.3) we find that

$$c_\nu^{(i}(r) = \sqrt{\frac{r^{n+2\nu}}{n+2\nu}} k_\nu^{(i} \quad (\nu = 1, 2, ...). \tag{4.4}$$

Hence for any $0 < r < R$

$$\sum_{i,\nu} |k_\nu^{(i)}(r)|^2 \frac{r^{n+2\nu}}{n+2\nu} = r^n \sum_{\nu=0}^{\infty} \left(\sum_{i=1}^{J(\nu)} \frac{|k_\nu^{(i)}(r)|^2}{n+2\nu} \right) r^{2\nu} < \infty$$

Using the Cauchy-Hadamard formula for the radius of the convergence of a power series we obtain

$$\limsup_{\nu\to\infty} \max_i \sqrt[\nu]{|k_\nu^{(i)}(y)|} \leq \limsup_{\nu\to\infty} \left(\sum_{i=1}^{J(\nu)} \frac{|k_\nu^{(i)}(r)|^2}{n+2\nu} \right)^{1/2\nu} \leq \frac{1}{r}$$

Since $0 < r < R$ is arbitrary then condition (4.1) holds, which was to be proved.

Sufficiency. If condition (4.1) holds then the Cauchy-Hadamard formula and the estimate $J(\nu) < const\, \nu^{n-2}$ implies that the series $\sum_{i,\nu} |k_\nu^{(i)}(r)|^2 \frac{r^{n+2\nu}}{n+2\nu}$ converges for any $0 < r < R$. The Riesz-Fisher theorem implies that there exists a section $\mathcal{F}$ (of the bundle $E_{|B_r}$) with the components from $S^2_{\Delta_n}(B_r)$ such that

$$\mathcal{F}(x) = \sum_{i,\nu} \sqrt{\frac{r^{n+2\nu}}{n+2\nu}} k_\nu^{(i} \sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h_\nu^{(i}(x) =$$

$$= \sum_{i,\nu} k_\nu^{(i} h_\nu^{(i}(x)$$

where the series converges in the norm of the space $L^2(E_{B_r})$. It is easy to see that in the ball Ω the section $\mathcal{F}$ coincides with $\mathcal{F}$. Therefore it is a harmonic extension of Green's integral $\mathcal{F}$ from D^+ to the whole domain D.

Now using Theorem 1.2 we conclude that Problem 1 is solvable. This proves the theorem. $\square$

Let us assume now that the surface S can be axtended smoothly to a neighbourhood of $\overline{D}$. Then we have

Corollary 4.2. *Let $S \in C^2$ and let f_0, $f_1 \in L^2(S)$. Then for Problem 1 to be solvable, it is necessary and sufficient that the series $\sum_{\nu,i} |a_\nu^{(i)}| \frac{R^{2\nu}}{n+2\nu}$ is convergent.*

Proof. Follows immediately from Theorems 3.6, 2.1 and formula (4.4). $\square$

§5. Carleman's formula

Bases with double orthogonality can be used to prove Carleman's formula for determination of a harmonic function f in D^- by its Cauchy data on S. To illustrate, let us consider a ball in $\mathbb{R}^n$.

For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x, y) = g(x - y) - g(y) + \sum_{\nu=1}^{N} \sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{n + 2\nu - 2} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}.$$

Proposition 5.1. *For any number $N = 1, 2, ...$, the kernel $\mathfrak{C}^{(N)}$ is harmonic with respect to x and y for all $y \neq 0$ off the diagonal $\{x = y\}$.*

Proof. Follows from the properties of the $g(x-y)$ and the polynomials $h_\nu^{(i)}(y)$. $\square$

Theorem 5.2. *For any harmonic function $f \in C_{loc}(D \cup S)$ whose restriction to S is summable there, the following formula holds*

$$f(x) = \lim_{N\to\infty} \int_S \left(f(y) \frac{\partial \mathfrak{C}^{(N)}(x, y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} \mathfrak{C}^{(N)}(x - y) \right) ds(y) \ (x \in D^-). \tag{5.1}$$

Proof. Let f_o and f_1 stands for the restrictions to S of f and its notmal derivative, respectively. Since f is a solution of Problem 1, it follows from Theorem 1.2 that $\mathcal{F}^+$ has harmonic extension to the ball D, say Φ. It is evident from Theorem 1.2 that the function $\hat{f} = \mathcal{F}^- - \Phi$ is also a solution of Problem 1. It is readily seen that in that case $\hat{f}$ coincides with f in D^-.

Further, for any $0 < r < R$ we have $\Phi \in h^2(B_r)$. Since any point $x \in D^-$ is also in some ball smaller than B_R, say B_r, it follows from formula (4.4) that

$$f(x) = \mathcal{F}^-(x) - \Phi(x) = \mathcal{F}^-(x) - \sum_{\nu,i} c_\nu^{(i)} Q_\nu^{(i)}(x) =$$

$$\mathcal{F}^-(x) - \sum_{\nu,i} a_\nu^{(i)} h_\nu^{(i)}(x) = \mathcal{F}^-(x) - \lim_{n\to\infty} \sum_{\nu=0}^{N} \sum_{i=1}^{J(\nu)} a_\nu^{(i)} h_\nu^{(i)}(x).$$

The limit on the right hand side exists in the norm of $L^2(B_r)$ in any ball B_r $(0 < r < R)$. In particular, by Stieltjes-Vitali theorem the convergence is uniform together with all derivatives on compact subsets of B_r.

Since the point zero is not on S, by assumption of the theorem, we have

$$f(x) = \int_S \left(f(y) \frac{\partial g(x-y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} g(x-y) \right) ds(y) -$$

$$f(x) - \int_S \left(f(y) \frac{\partial g(y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} g(y) \right) ds(y) +$$

$$\lim_{N\to\infty} \sum_{\nu=0}^{N} \sum_{i=1}^{J(\nu)} \int_S \left(f(y) \frac{\partial}{\partial n_y} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} - \frac{\partial f(y)}{\partial n_y} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \right) ds(y) \frac{h_\nu^{(i)}(x)}{n+2\nu-2}$$

$$= \lim_{N\to\infty} \int_S \left(f(y) \frac{\partial \mathfrak{C}^{(N)}(x,y)}{\partial n_y} - \frac{\partial f(y)}{\partial n_y} \mathfrak{C}^{(N)}(x-y) \right) ds(y) \ (x \in D^-).$$

□

Remark 5.3. As one can see from the proof of Theorem 5.2, the convergence of the limit in (5.1) is uniform on compact subsets of the domain D^- together with all its derivatives.

Carleman's formula was established in [6] for specific choice of D^-, bounded by part of the surface of the cone $\mathcal{K}$ and a smooth piece of S in the interior of $\mathcal{K}$.

In conclusion I would like to thank Prof. N.N. Tarkhanov for his guidance in the preparation of this paper.

References

1. J. Hadamard, *Le problème de Cauchy et les equations aux derivées partielles linéares hyperboliques*, Gauthier - Villars, Paris, 1932.
2. S.N. Mergeljan, *Harmonic approximation and approximate solution of the Cauchy problem for Lapla- ce's equation*, Uspechi. Mat. Nauk **II 5** (1956), 3–26. in Russian
3. M.M. Lavrent'ev, *On the Cauchy problem for Laplace's equation*, Izvestija AN SSSR. Ser. mat. **20** (1956), 819-842. in Russian
4. E.M. Landis, *On some properties of solution of elliptic equations*, Dokl. AN SSSR **107 5** (1956), 640–643. in Russian
5. V.G. Maz'ya and V.P. Havin, *On the solutions of the Cauchy problem for Laplace's equation (uniqueness, normality, approximation)*, Trudy Mosk. Mat. Ob-va **307** (1974), 61–114; English Transl. in Transactions of the Moscow Math. Soc. **30** (1974), 65–118.
6. S.J. Jarmuhamedov, *On the Cauchy problem for Laplace's equation*, Dokl AN SSSR **235** (21977), 281–283; English transl. in Sov. Math. Dokl. **18 4** (1977), 939–943.
7. N.N. Tarkhanov, *Solvability criterion for the ill-posed Cauchy problem for elliptic systems*, Dokl. AN SSSR **308 3** (1989), 531–534. in Russian
8. A.A. Shlapunov and N.N. Tarkhanov, *On the Cauchy problem for holomorphic functions of Lebesgue class L^2 in domain*, Siberian math. journal **33 5** (1992).
9. N.N. Tarkhanov, *Laurent series for solutions of elliptic systems*, Nauka, Novosibirsk, 1991. in Russian
11. E.B. Fabes, M. Jodeit, N.M. Rivière, *Potential techniques for boundary value problems on C^1 domains*, Acta Math. **141 3–4** (1978), 167–186.

11. S.L. Sobolev, *Introduction to the Theory of Cubature formulae*, Nauka, Moscow, 1974. in Russian
12. L.A. Aizenberg and A.M. Kytmanov, *On the possibility of holomorphic extension into a domain of functions defined on a connected piece of its boundary*, Mat. Sbornik **182 4** (1991), 490–507; English transl. in Math. USSR Sbornik, **72 2** (1992).
13. V.G. Maz'ya and V.P. Appriximation by harmonic functions in the mean, *Studies in Linear Operators and Function Theory (Proc. Sci. Seminars, Leningrad Branch of Math. Inst.)*, Nauka, Leningrad, 1972, pp. 91–105. in Russian

Krasnoyarsk State University, pr. Svobodny, 79, 660036, Krasnoyarsk, Russia