SPECTRAL DECOMPOSITION OF GREEN'S INTEGRALS AND EXISTENCE OF $W^{s,2}$-SOLUTIONS OF MATRIX FACTORIZATIONS OF THE LAPLACE OPERATOR IN A BALL

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Abstract. We prove that Green's integral $\mathcal{G}$ for a ball $B_R \subset \mathbb{R}^n$, associated to a matrix factorization P of the Laplace operator, defines a bounded linear selfadjoint operator in the Hilbert space of harmonic $W^{s,2}$-functions in B_R ($s \geq 0$), with the spectrums belonging to the interval $[0,1]$ on the real axis. Using this fact, we obtain a formula for solutions of the equation $Pu = f$ in B_R with $W^{m,2}$-datum, and a criterion for the existence of $W^{s,2}$-solutions for such a datum ($m \geq 0$, $0 \leq s \leq m+1$).

Let P be a matrix factorization of the Laplace operator Δ_n in $\mathbb{R}^n$ ($n \geq 2$), i.e. a homogeneous first order partial linear differential operator (with constant coefficients) such that

$$-P^* P = \Delta_n I_k$$

where I_k is the unit $(k \times k)$-matrix and P^* is the formal adjoint of P. We have

$$P = \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i}, \quad P^* = -\sum_{i=1}^{n} P_i^* \frac{\partial}{\partial x_i}$$

with complex valued $(l \times k)$-matrices P_i ($l \geq k$).

In this paper we are interested in finding $W^{s,2}$-solutions of the equation

$$Pu = f \text{ } in \text{ } B_R, \tag{0.1}$$

where B_R is the ball with centre at the origin and radius $0 < R < \infty$.

With the purpose we study in §1 spectrum of operators defined by Green's integral, associated with P and the standard fundamental solution of the Laplace operator $\mathbb{R}^n$, in the Hilbert spaces $h^{s,2}(B_R)$ of harmonic $W^{s,2}$-functions in B_R ($s \geq 0$). Following the approach of A.V. Romanov (cf. [Rom1]), we prove that Green's integrals define bounded linear selfadjoint operators in $h^{s,2}(B_R)$, with the spectrums belonging to the interval $[0,1]$ on the real axis. For this, however, we need to define a special Hilbert structure in $h^{s,2}(B_R)$ for non-integral s.

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As a corollary, in §2 we prove Theorem on Iterations of Green's integrals in the Hilbert spaces $h^{s,2}(B_R)$ and obtain a formula for $W^{s,2}$-solutions of the equation $Pu = f$ in B_R, with $W^{m,2}$-data, whenever they exists (cf. [Rom2], [Sh2] and [NaSh]).

Then, using the spectral decomposition of the operators defined by Green's integrals, we obtain in §3 necessary and sufficient conditions of the fact that for every $f \in W^{m,2}(B_R)$ $(m \geq 0)$, satisfying compatibility conditions in B_R, there exists a solution $u \in W^{s,2}(B_R)$ with $Pu = f$ and $0 \leq s \leq m+1$, and obtain a formula for solutions of the equation $Pu = f$ in B_R with $W^{m,2}$-data.

The formulae produce rather explicitely a way to obtain a solution of the equation $Pu = f$ by successive approximations.

Though the system P seems to be rather simple, there are no results about existence of its $W^{s,2}$-solution for $W^{m,2}$-data unless the well known examples such as the operators connected with the De Rham complex or the Dolbeault complex (in particular, the gradient operator or the Cauchy-Riemann system). Of course, the results about solvability of systems with constant coefficients in convex domains (see, for example, [AnNa]) and theorems on the regularity of solutions of elliptic systems imply that for every $f \in W^{m,2}(B_R)$, satisfying the compatibility conditions in B_R there exists a solution $u \in W^{m+1,2}_{loc}(B_R)$ with $Pu = f$. However the example of the overdetermined Cauchy-Riemann system shows that we should expect a finite loss of the global regularity of solutions (see, for example, Example 7.4 in [NaSh] and Corollary 3.5 and Example 3.7 below).

In the case of a determined matrix factorization of the Laplace operator in $\mathbb{R}^n$ (i.e. $l = k$), for every $f \in W^{m,2}(B_R)$ there exist a solution $u \in W^{m+1,2}(B_R)$ with $Pu = f$. So, throughout the paper we will be concentrated on the case where the operator P is overdetermined, i.e. $l > k$ (though, the case $l = k$ is formally permitted).

§1. Spectrum of Green's integrals in $h^{s,2}(B_R)$

We denote by B_R the ball in $\mathbb{R}^n$ with centre at zero and radius $0 < R < \infty$, by σ_n the area of the unit sphere ∂B_1 and by $d\sigma$ the standard volume form on ∂B_R.

Let $[W^{s,2}(B_R)]^k$ $(s \geq 0)$ be the Hilbert space of k-vectors of complex valued functions, having the components in the Sobolev space $W^{s,2}(B_R)$ and let $[h^{s,2}(B_R)]^k$ be its closed subspace formed by vector functions with harmonic components.

For $s \in \mathbb{Z}_+$, we provide $[W^{s,2}(B_R)]^k$ with the scalar product

$$(u, v)_{W^{s,2}(B_R)} = \int_{|y| \leq R} \sum_{j=1}^{k} \sum_{|\alpha| \leq s} (D^\alpha u_j)(y) \overline{(D^\alpha v_j)(y)} dy \quad (u, v \in [W^{s,2}(B_R)]^k).$$

Hence $[h^{s,2}(B_R)]^k$ is a Hilbert space with the induced from $[W^{s,2}(B_R)]^k$ Hilbert structure.

It is known that for $u \in [h^{s,2}(B_R)]^k$ there exists weak boundary values $(D^\alpha u)_{|\partial B_R}$ belonging to the Sobolev space $[W^{s-|\alpha|-1/2,2}(\partial B_R)]^k$ (see, for example [ShTa], Theorem 4.4). Then, for $s = N - 1/2$ $(N \in \mathbb{N})$ we provide $[h^{s,2}(B_R)]^k$ with the scalar product

$$(u, v)_{h^{s,2}(B_R)} = (u, v)_{W^{[s],2}(\partial B_R)} = \int_{|y|=R} \sum_{j=1}^{k} \sum_{|\alpha| \leq [s]} (D^\alpha u_j)(y) \overline{(D^\alpha v_j)(y)} d\sigma(y),$$

where $u, v \in [h^{s,2}(B_R)]^k$ and $[s]$ is the integral part of s. It is not difficult to see that in this case $[h^{s,2}(B_R)]^k$ is a Hilbert space too with the topology equivalent to the one induced from $[W^{s,2}(B_R)]^k$.

For example, $[h^{1/2,2}(B_R)]^k$ is the Hardy space of harmonic k-vector complex valued functions in B_R, $[h^{1/2,2}(B_R)]^k \subset [h^{0,2}(B_R)]^k$ and these spaces are not equal.

For other non-integral s we will define a special Hilbert structure in $[h^{s,2}(B_R)]^k$ later.

Now, for a vector $u \in [h^{0,2}(B_R)]^k$ we denote by $\mathcal{G}u$ its Green's integral in B_R associated with the operator P and the standard fundamental solution $\varphi_n(x-y)$ of the Laplace operator in $\mathbb{R}^n$:

$$\mathcal{G}u(x) = \frac{1}{R\sigma_n} \lim_{r\to R} \int_{|y|=r} \left(\sum_{i=1} P_i^*(y_i - x_i) \right) \left(\sum_{j=1} P_j y_j \right) \frac{u(y)}{|y-x|^n} d\sigma(y) \ (|x| \neq R).$$

It follows from [ReSz] (2.3.2.5), that the integral $\mathcal{G}$ defines a bounded linear operators

$$\mathcal{G}_s : [h^{s,2}(B_R)]^k \to [h^{s,2}(B_R)]^k.$$

In this section we are interested in the spectrum of the operators $\mathcal{G}_s$ in $[h^{s,2}(B_R)]^k$. We follow the approach of Romanov for the Martinelli-Bochner integral and $s = 1/2$ (see [Rom1]). For this purpose we will use the following lemma.

Lemma 1.1. *For every homogeneous harmonic polynomial h_ν of degree $\nu \geq 0$ in $\mathbb{R}^n$ we have*

$$\mathcal{G}h_0 = h_0, \ \mathcal{G}h_\nu(x) = h_\nu(x) - \left(\sum_{i=1} P_i^* x_i \right) \frac{(Ph_\nu)(x)}{n+2\nu-2} \quad (\nu \geq 1). \tag{1.1}$$

Proof. It is proved by direct calculations. $\square$

We extract from Lemma 1.1 an information about the spectrum of the operator $\mathcal{G}_0$.

Lemma 1.2. *There exists an orthonormal basis $\{h_\nu^{(i,R)}\}$ ($1 \leq i \leq \frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$, $\nu \geq 0$) in $[h^{0,2}(B_R)]^k$ consisting of homogeneous harmonic polynomials with*

$$\mathcal{G}h_\nu^{(i,R)} = \lambda_\nu^{(i,R)} h_\nu^{(i,R)}, \ 0 \leq \lambda_\nu^{(i,R)} \leq 1.$$

Proof. Let us denote by $S_k(\nu)$ the vector space of all the k-vectors of homogoneous harmonic polynomials of degree ν. It is a finite dimensional vector space with $dim S_k(\nu) = \frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$. Lemma 1.1 implies that $\mathcal{G}_{|S_k(\nu)} : S_k(\nu) \to S_k(\nu)$ is a bounded linear operator.

Since $S_k(\nu)$ is finite dimensional, it is a (complex) Hilbert space with the scalar product $(.,.)_{L^2(B_R)}$. On the other hand, due to Lemma 1.1 and Stokes' formula,

$$(\mathcal{G}h_\nu, g_\nu)_{L^2(B_R)} = (h_\nu, g_\nu)_{L^2(B_R)} - \frac{R^2\,(Ph_\nu, Pg_\nu)_{L^2(B_R)}}{(n+2\nu-2)(n+2\nu)}. \tag{1.2}$$

In particular, this means that the operator $\mathcal{G}_{|S_k(\nu)}$ is selfadjoint. Hence we conclude that *spectr* $\mathcal{G}_{|S_k(\nu)} \subset [-m, m]$, where m is the operator norm of $\mathcal{G}_{|S_k(\nu)}$.

Since the space $S_k(\nu)$ is finite dimensional, there exist $\frac{k(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ eigenvectors of the operator $\mathcal{G}_{|S_k(\nu)}$ (corresponding to eigenvalues $\lambda_\nu^{(i,R)}$), which form an orthogonal (with respect to $(.,.)_{L^2(B_R)}$) basis in $S_k(\nu)$.

For $\nu = 0$, $h_0^{(i)} = \sqrt{R^n V_n} 1_i$, $\lambda_0^{(i)} = 1$ $(1 \le i \le k)$, where 1_i is k-vector with components $1_i^j = \delta_{ij}$ and V_n is the volume of the unit ball in $\mathbb{R}^n$.

Let $\nu \ge 1$ ($\nu \ge 2$ if $n = 2$). Because $S_k(\nu)$ is finite dimensional, it is a (complex) Hilbert space with the scalar product (cf. [Sh2] and [NaSh])

$$H_P(h_\nu, g_\nu) = \int_{B_R} (Pg_\nu)^*(y)(Ph_\nu)(y)dy + \int_{\mathbb{R}^n \setminus B_R} (PS(g_\nu))^*(y)(PS(h_\nu))(y)dy \ (h_\nu \in S_k(\nu)) \tag{1.3}$$

where $S(h_\nu) = \frac{R^{n+2\nu-2} h_\nu(x)}{|x|^{n+2\nu-2}}$ is a harmonic function outside of the ball B_R with zero at infinity and $S(h_\nu) = h_\nu$ on ∂B_R. Then, by the Stokes' formula, we have (cf. [NaSh])

$$H_P(\mathcal{G}h_\nu, g_\nu) = \int_{\mathbb{R}^n \setminus B_R} (PS(g_\nu)^*(y)PS(h_\nu))(y)dy, \tag{1.4}$$

and, in particular, $0 \le H_P(\mathcal{G}h_\nu, h_\nu) \le H_P(h_\nu, h_\nu)$. Hence we conclude that $0 \le \lambda_\nu^{(i,R)} \le 1$ $(1 \le i \le dim S_k(\nu))$.

For the case $n = 2$, $\nu = 1$, we have $\lambda_1^{(i)} = 1 - \mu_1^{(i)}/2$, where $\mu_1^{(i)}$ are eigenvalues of the symmetric block-matrix $Q = \begin{pmatrix} I_k & P_1^* P_2 \\ P_2^* P_1 & I_k \end{pmatrix}$, $\|Q\| = max|q_{mN}| \le 1$, i.e. $0 \le \lambda_1^{(i)} \le 1$.

It is known (see, for example, [Sh1]) that it is possible to choose in the space $[h^{0,2}(B_R)]^k$ a basis $\{\tilde{h}_\nu^{(i)}\}$ with $\tilde{h}_\nu^{(i)} \in S_k(\nu)$ and $1 \le i \le dim S_k(\nu)$. Therefore, because spherical harmonics of different degrees of homogeneity are orthogonal in $[L^2(B_R)]^k$, we can choose an orthonormal basis $\{h_\nu^{(i,R)}\}$ $(\nu \ge 0,\ 1 \le i \le dim S_k(\nu))$ in $[h^{0,2}(B_R)]^k$, consisting of the eigenfunctions of the operator $\mathcal{G}$. $\square$

Lemma 1.3. *$h_\nu^{(i)} = h_\nu^{(i,1)} = \sqrt{R^{n+2\nu}} h_\nu^{(i,R)}$ and $\lambda_\nu^{(i)} = \lambda_\nu^{(i,1)} = \lambda_\nu^{(i,R)}$; $\lambda_\nu^{(i)} = 1$ if and only if $Ph_\nu^{(i)} = 0$.*

Proof. This is an immediate sequence of the homogeneity of the polynomials and formulae (1.3), (1.4). $\square$

Because the polynomials $h_\nu^{(i)}$ are homogeneous, it is not difficult to obtain from Green's formula for harmonic functions the following lemma (cf. [Sh1]).

Lemma 1.4. *The system $\{h_\nu^{(i)}\}$ is an orthogonal basis in $[h^{s,2}(B_R)]^k$ ($s = (N-1)/2$, $N \in \mathbb{N}$). Moreover there exist constants $C_1(s,n)$, $C_2(s,n) > 0$ such that*

$$C_1(s,n)\|h_\nu^{(i)}\|^2_{W^{s,2}(B_R)} \le \nu^{2s}\|h_\nu^{(i)}\|^2_{L^2(B_R)} \le C_2(s,n)\|h_\nu^{(i)}\|^2_{W^{s,2}(B_R)}$$

for every $\nu \ge 0$, $1 \le i \le dim S_k(\nu)$.

Proof. The orthogonality and estimates can be proved by direct calculations, using the homogeneity of the polynomials.

The system $\{h_\nu^{(i)}\}$ is complete in $[h^{s,2}(B_R)]^k$ because it is complete in $[h^{0,2}(B_R)]^k$ and orthogonal in $[h^{s,2}(B_R)]^k$. $\square$

Now, for $s \geq 0$ ($s \neq (N-1)/2$, $N \in \mathbb{N}$) we provide the space $[h^{s,2}(B_R)]^k$ with the Hermitian form

$$(u,v)_{h^{s,2}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} C_\nu^{(i)}(u)\overline{C_\nu^{(i)}(v)}\nu^{2s} \ (u,v \in [h^{s,2}(B_R)]^k),$$

where $C_\nu^{(i)}(u)$ are the Fourier coefficients of the vector-function u with respect to the orthonormal basis $\{h_\nu^{(i)}\}$ in $[h^{0,2}(B_R)]^k$.

Proposition 1.5. *The Hermitian form $(.,.)_{h^{s,2}(B_R)}$ ($s \geq 0$) is a scalar product in the space $[h^{s,2}(B_R)]$ defining a topology, equivalent to the original one. Moreover, the system $\{h_\nu^{(i)}\}$ is an orthogonal basis in $[h^{s,2}(B_R)]^k$ and there exist constants $C_1(s,n)$, $C_2(s,n) > 0$ such that*

$$C_1(s,n)\|h_\nu^{(i)}\|^2_{W^{s,2}(B_R)} \leq \nu^{2s}\|h_\nu^{(i)}\|^2_{L^2(B_R)} \leq C_2(s,n)\|h_\nu^{(i)}\|^2_{W^{s,2}(B_R)}$$

for every $\nu \geq 0$, $1 \leq i \leq dim S_k(\nu)$.

Proof. For $s = (N-1)/2$, $N \in \mathbb{N}$ the statement was proved in Lemma 1.4. In order to prove it for a number $s \geq 0$ ($s \neq (N-1)/2$, $N \in \mathbb{N}$), it is sufficient to consider 2 interpolation couples: $[h^{[s],2}(B_R)]^k$, $[h^{[s]+1,2}(B_R)]^k$ and $l_2([s])$, $l_2([s]+1)$, where $[s]$ is the integral part of s and, for $r \geq 0$, $l_2(r) = \{\{K_\nu\}_{\nu=0}^\infty : \sum_{\nu=1}^\infty |K_\nu|^2\nu^{2r} < \infty\}$, and then to use the standard interpolation arguments (see, for example, [Tr], 4.1–4.4 and 1.18.2). $\square$

The following theorem was proved for the Martinelli-Bochner integral in the ball and $s = 1/2$ in Romanov ([Rom1]), for the Martinelli-Bochner integral in a domain with connected boundary and $s = 1$ in Romanov ([Rom2]). For Green's integrals, associated with the matrix factorization of the Laplace operator in a domain with smooth boundary, and $s = 1$ it was proved in [Sh2] and for Green's integrals, associated with an elliptic (overdetermined) system of order $p \geq 1$ and special fundamental solutions, and $s = p$ its generalization was obtained in [NaSh].

Theorem 1.6. *The operator $\mathcal{G}_s : [h^{s,2}(B_R)]^k \to [h^{s,2}(B_R)]^k$ is a bounded linear selfadjoint operator with spectr $\mathcal{G}_s \subset [0,1]$ ($s \geq 0$).*

Proof. According to Lemmata 1.2, 1.3 and Proposition 1.5, for a vector $u \in [h^{s,2}(B_R)]^k$, we have

$$(\mathcal{G}u,u)_{h^{s,2}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} \lambda_\nu^{(i)}|C_\nu^{(i)}(u)|^2\|h_\nu^{(i)}\|^2_{h^{s,2}(B_R)} \geq 0,$$

$$\|\mathcal{G}u\|^2_{h^{s,2}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} (\lambda_\nu^{(i)})^2|C_\nu^{(i)}(u)|^2\|h_\nu^{(i)}\|^2_{h^{s,2}(B_R)} \leq \|u\|^2_{h^{s,2}(B_R)}$$

because $0 \leq \lambda_\nu^{(i)} \leq 1$ for the eigenvalues of the operators $\mathcal{G}_s$. Here $C_\nu^{(i)}(u)$ are the Fourier coefficients of u with respect to the orthogonal system $\{h_\nu^{(i)}\}$. Therefore $\mathcal{G}_s$ is a selfadjoint operator with *spectr* $\mathcal{G}_s \subset [0,1]$. $\square$

Because of the orthogonality and the completeness of the system $\{h_\nu^{(i)}\}$, $0 \leq \lambda_\nu^{(i)} \leq 1$ are the only eigenvalues of the operators $\mathcal{G}_s$.

Example 1.7. Let $P = \nabla_n$ $(n \geq 2)$ be the gradient operator in $\mathbb{R}^n$ $(l = n,\ k = 1)$. Then, due to Lemma 1.1 and Euler formula for homogeneous functions, for every homogeneous harmonic polynomial h_ν we have

$$\mathcal{G} h_\nu = \frac{n+\nu-2}{n+2\nu-2} h_\nu .$$

Arguing as before, we obtain that the multiplicity of the eigenvalue $\frac{1}{2} < \frac{n+\nu-2}{n+2\nu-2} \leq 1$ is $\frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!} < \infty$ and that *spectr* $\mathcal{G}$ consists of the eigenvalues $\frac{n+\nu-2}{n+2\nu-2}$ and the limit point $1/2$, if $n \geq 3$.

In the degenerate case $n = 2$ the spectrum *spectr* $\mathcal{G}$ consists only of two eigenvalues: $1/2$ (eigenvalue of the infinite multiplicity corresponding to $\nu > 0$) and simple eigenvalue 1 (corresponding to $\nu = 0$), i.e $\mathcal{G}$ is not compact. $\square$

Example 1.8. Let $P = 2\overline{\partial}$ be the (doubled) Cauchy-Riemann system in $\mathbb{C}^m$ $(m \geq 2)$ written in the complex form with the complex coordinates z_j, $\overline{z}_j$ $(1 \leq j \leq m)$. Then $n = 2m$, $l = m$, $k = 1$ and $\mathcal{G}$ is the Martinelli-Bochner integral.

Romanov (see [Rom1]) studied the spectrum of the operator $\mathcal{G}$ in the Hardy spaces $H^2(B_1)$ $(\cong h^{1/2,2}(B_R))$ and $H^2(\mathbb{C}^m \backslash B_1)$ $(\cong h^{1/2,2}(\mathbb{C}^m \backslash B_R))$. He proved that harmonic homogeneous polynomials

$$h_{rt} = \sum_{|\alpha|=s} \sum_{|\beta|=t} c_{\alpha\beta} z^\alpha \overline{z}^\beta$$

with multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$ and degree of the homogeneity $\nu = r + t$, are the eigenvalues of the operator $\mathcal{G}$:

$$\mathcal{G} h_{rt} = \frac{m+r-1}{m+r+t-1} h_{rt},$$

and that we can always choose an orthogonal basis $\{\widetilde{h}_{st}\}$ $(r \geq 0,\ t \geq 0)$ in $H^2(B_1)$ $(\cong h^{1/2,2}(B_R))$ consisting of polynomials of the type h_{rt}.

One easily checks that this implies that all rational numbers of the interval $[0, 1]$ are eigenvalues of infinite multiplicity of the Martinelli -Bochner integral $\mathcal{G}$, and that *spectr* $\mathcal{G}_s$ = *spectr* $\mathcal{G}_s^{(c)} = [0, 1]$. In particular, the operators $\mathcal{G}_s$ and $\mathcal{G}_s^{(c)}$ are not compact.

In the degenerate case $m = 1$ we have $n = 2$, $l = 1$, $k = 1$ and $\mathcal{G}$ is the Cauchy integral. The spectrum *spectr* $\mathcal{G}$ consists only of two eigenvalues (both are of infinite multiplicity): 1 (eigenvalue corresponding to z^ν, $\nu \geq 0$), and 0 (eigenvalue corresponding to $\overline{z}^\nu$, $\nu > 0$), i.e $\mathcal{G}$ is not compact. $\square$

§2. Theorem on iterations of Green's integrals

In this section we will use the information about the spectrum of the operators $\mathcal{G}_s : [h^{s,2}(B_R)]^k \to [h^{s,2}(B_R)]^k$ $(s \geq 0)$ to obtain Theorem on Iterations of Green's integrals (cf. [Rom1], [Rom2], [Sh2], and [NaSh]).

It is known that for $u \in [h^{s,2}(B_R)]^k$ $(s \geq 0)$ there exists weak boundary value $Pu_{|\partial B_R}$ belonging to the Sobolev space $[W^{s-3/2,2}(\partial B_R)]^k$. Let us denote by $\tau P u$ the single layer potential:

$$(\tau P u)(x) = \frac{1}{R} \lim_{r \to R} \int_{|y|=r} \left(\sum_{j=1} P_j^* y_j \right) \varphi_n(x-y)(Pu)(y) d\sigma(y).$$

By Stokes' formula we have

$$(\mathcal{G}u)(x) + (\tau P u)(x) = \begin{cases} u(x), x \in B_R, \\ 0, x \in X\backslash\overline{B}_R \end{cases} \tag{2.1}$$

for every $u \in [h^{s,2}(B_R)]^k$. Therefore the integral τP defines linear bounded operators $\tau_s P$ from $[h^{s,2}(B_R)]^k$ to $[h^{s,2}(B_R)]^k$.

In particular, it is possible to consider iterations $\mathcal{G}^\nu = \mathcal{G} \circ \mathcal{G} \circ \cdots \circ \mathcal{G}$, $(\tau P)^\nu = (\tau P) \circ (\tau P) \circ (\dots) \circ (\tau P)$ ($\nu \geq 1$ times) of the integrals $\mathcal{G}$ and τP in these spaces.

Let $S_P^{s,2}(B_R)$ be the closed subspace of $[h^{s,2}(B_R)]^k$ consisting of solutions of the system $Pu = 0$ in B_R. Then $\Pi(S_P^{s,2}(B_R))$ stand for the orthogonal projections from $[h^{s,2}(B_R)]^k$ to $S_P^{s,2}(B_R)$.

Since $[h^{s,2}(B_R)]^k$ is topologically isomorphic to $[h^{s,2}(\mathbb{R}^n\backslash B_R)]^k$ ($n \geq 3$), we associate $u \in [h^{s,2}(B_R)]^k$ a (unique) vector function $S(u) \in [h^{s,2}(\mathbb{R}^n\backslash B_R)]^k$ with $u = S(u)$ on ∂B_R.

In the case where $n = 2$ we associate $u \in [h^{s,2}(B_R)]^k$ a (unique) vector function $S(u)$, harmonic in $\mathbb{R}^2\backslash B_R$, regular at infinity with respect to φ_n (see [Ta]) and such that $u = S(u)$ on ∂B_R.

Then we can consider $S_P^{s,2}(\mathbb{R}^n\backslash B_R) = \{u \in [h^{s,2}(B_R)]^k : \ PS(u) = 0 \ in \ \mathbb{R}^n\backslash B_R\}$ as a closed subspace of $[h^{s,2}(B_R)]^k$ and $\Pi(S_P^{s,2}(\mathbb{R}^n\backslash B_R))$ stands for the orthogonal projection from $[h^{s,2}(B_R)]^k$ to $S_P^{s,2}(\mathbb{R}^n\backslash B_R)$.

For the Martinelli-Bochner intefgral and $s \in \mathbb{Z}_+$ this fact was mentioned in [Ky].

Theorem 2.1 (on Iterations). *Let $s \geq 0$. Then*

$$\lim_{\nu\to\infty} \mathcal{G}_s{}^\nu = \Pi(S_P^{s,2}(B_R)), \quad \lim_{\nu\to\infty} (\tau_s P)^\nu = \Pi(S_P^{s,2}(\mathbb{R}^n\backslash B_R))$$

in the strong operator topology in $[h^{s,2}(B_R)]^k$.

Proof. Theorem 1.6 and (2.1) imply that the operators $\tau_s P$, $\mathcal{G}_s$ are selfadjoint operators in $[h^{s,2}(B_R)]^k$ and that $0 \leq \mathcal{G}_s \leq Id, 0 \leq \tau_s P \leq Id$ (where Id stands for the identity operator on $[h^{s,2}(B_R)]^k$). Moreover, Lemma 1.3 and Proposition 1.5 yield that for every $u \in [h^{s,2}(B_R)]^k$ we have

$$\mathcal{G}^\nu u = \sum_{\mu=0}^{\infty} \sum_{i=1}^{dim S_k(\mu)} (\lambda^{(i)})^\nu C_\mu^{(i)}(u) h_\mu^{(i)}, \tag{2.2}$$

$$(\tau P)^\nu u = \sum_{\mu=0}^{\infty} \sum_{i=1}^{dim S_k(\mu)} (1 - \lambda^{(i)})^\nu C_\mu^{(i)}(u) h_\mu^{(i)}, \tag{2.3}$$

where $C_\mu^{(i)}(u)$ are the Fourier coefficients of $u \in [h^{s,2}(B_R)]^k$ with respect to the orthogonal basis $\{h_\mu^{(i)}\}$ in $[h^{0,2}(B_R)]^k$.

Because $0 \leq \mathcal{G}_s \leq Id$, $0 \leq \tau_s P \leq Id$, the series in (2.3) (2.4) converge uniformly with respect to ν. Hence, passing to the limit in (2.2) and (2.3) and using Lemma 1.3, (2.1) and (1.4), one obtains the statement of the theorem. □

Remark 2.2. If P is the gradient operator ∇_n in $\mathbb{R}^n$ or the doubled Cauchy-Riemann system $\overline{\partial}$ in $\mathbb{C}^n$ then $S_{\nabla_n}^{s,2}(\mathbb{R}^n\backslash B_R) = 0$, $S_{\overline{\partial}}^{s,2}(\mathbb{C}^n\backslash B_R) = 0$. However it is not true for all overdetermined matrix factorizations of the Laplace operator. For example, if $n=3$, $l=4$, $k=3$ and

$$P = \begin{pmatrix} \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \end{pmatrix},$$

then it is easy to check that the vector $x \in [h^{s,2}(B_R)]^3$ belongs to $S_P^{s,2}(\mathbb{R}^3 \backslash B_R)$ with $S(x) = \frac{R^3\, x}{|x|^3}$. Hence $\ker \mathcal{G}_s = S_P^{s,2}(\mathbb{R}^3\backslash B_R) \neq 0$.

Corollary 2.3. *For every $u \in [h^{s,2}(B_R)]^k$ we have*

$$u = \lim_{\nu\to\infty} G^\nu u + \sum_{\mu=0}^{\infty} G^\mu \tau P u = \lim_{\nu\to\infty} (\tau P)^\nu u + \sum_{\mu=0}^{\infty} (\tau P)^\mu (\mathcal{G} u), \tag{2.4}$$

where the limits and the series converge in the $[W^{s,2}(B_R)]^k$-norm.

Proof. Follows from formula (2.1) and Theorem 2.1. □

Let us consider, for $s \geq m \geq s-1 \geq 0$ the linear closed densely defined operator

$$P_{s,m} : [W^{s,2}(B_R)]^k \to [W^{m,2}(B_R)]^l.$$

And let now $dom\, P_{s,m} = \{u \in [W^{s,2}(B_R)]^k : (Pu) \in [W^{m,2}(B_R)]^l\}$. It is easy to see that $dom\, P_{s,m}$ is a Hilbert space with the scalar product

$$(u,v)_{W^{s,2}(B_R)} + (Pu, Pv)_{W^{m,2}(B_R)} \ (u, v \in dom\, P_{s,m}).$$

Let T be the the following integral:

$$Tf(x) = \frac{1}{\sigma_n} \int_{B_R} \left(\sum_{j=1}^n P_j^*(y_j - x_j) \right) \frac{f(y)}{|y-x|^n} dy \ (f \in [L^2(B_R)]^l).$$

The integral T defines bounded linear operators $T_m : [W^{m,2}(B_R)]^l \to [W^{m+1,2}(B_R)]^k$ (see [NaSh]). Hence the composition TP defines a bounded linear operator $T_m P_s : dom\, P_{s,m} \to [W^{m+1,2}(B_R)]^k$.

Now we can define an extension $\widetilde{\mathcal{G}}_s$ of the operator $\mathcal{G}_s : [h^{s,2}(B_R)]^k \to [h^{s,2}(B_R)]^k$ for $u \in dom\, P_{s,m}$. Indeed, if $u \in dom\, P_{s,m}$ then there exists a sequence $u_N \in [C^s(\overline{B}_R)]^k$ such that $\lim_{N\to\infty} \|u - u_N\|_{|W^{s,2}(B_R)} + \|Pu - Pu_N\|_{|W^{m,2}(B_R)} = 0$. Then, for $u \in dom\, P_{s,m}$, we set

$$\widetilde{\mathcal{G}}_s u = \lim_{N\to\infty} \frac{1}{R\sigma_n} \int_{|y|=R} \left(\sum_{i=1} P_i^*(y_i - x_i) \right) \left(\sum_{j=1} P_j y_j \right) \frac{u_N(y)}{|y-x|^n} d\sigma(y).$$

Stokes' formula and the continuity of the operator T_m imply that, for $u \in dom\, P_s$, $\widetilde{\mathcal{G}}_s u = \lim_{N\to\infty}(u_N - TPu_N) = u - (TP)_s u$, i.e. the operator $\widetilde{\mathcal{G}}_s$ is well defined and does not depend on the choice of the sequence u_N. Moreover, it follows from Stokes' formula, that $TPu = \tau P u$ for $u \in [h^{s,2}(B_R)]^k \cap dom\, P_{s,m}$. Hence (2.1) implies that $\mathcal{G} u = \widetilde{\mathcal{G}} u$.

Now Theorem 2.1 implies the folowing result (similar to Corollary 2.3).

Corollary 2.4. *For every $u \in dom\, P_{s,m}$ we have*

$$u = \lim_{\nu\to\infty} \tilde{\mathcal{G}}^{\nu} u + \sum_{\mu=0}^{\infty} \tilde{\mathcal{G}}^{\mu} TPu = \lim_{\nu\to\infty} (TP)^{\nu} u + \sum_{\mu=0}^{\infty} (TP)^{\mu} (\tilde{\mathcal{G}} u),$$

where the limits and the series converge in the $[W^{s,2}(B_R)]^k$-norm.

Now we obtain a formula for solutions of $Pu = f$ in B_R whenever they exists in $dom\, P_{s,m}$.

Corollary 2.5. *Let $f \in [W^{m,2}(B_R)]^l$ such that $Pv = f$ in B_R with $v \in dom\, P_{s,m}$ $(0 \le s-1 \le m \le s)$. Then the series*

$$u = \sum_{\mu=0}^{\infty} G^{\mu} Tf = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1, Ph_{\nu}^{(i)} \neq 0}^{dim S_k(\nu)} \frac{(n+2\nu-2)(n+2\nu)\, C_{\nu}^{(i)}(\tilde{\mathcal{G}} Tf)}{\|Ph_{\nu}^{(i)}\|^2_{L^2(B_1)}} h_{\nu}^{(i)}, \quad (2.6)$$

where

$$C_{\nu}^{(i)}(\tilde{\mathcal{G}} Tf) =$$

$$= \frac{1}{R\sigma_n} \frac{1}{(n+2\nu)(n+2\nu-2)} \int_{|y|=R} \left(\sum_{m=1}^{n} P_m^* \frac{\partial}{\partial y_m} \frac{\overline{h_{\nu}^{(i)}(y)}}{|y|^{n+2\nu-2}} \right) \left(\sum_{j=1}^{n} P_j y_j \right) (Tf)(y) d\sigma(y)$$

are the Fourier coefficients of the vector $\tilde{\mathcal{G}} Tf$ with respect to the orthogonal basis $\{h_{\nu}^{(i)}\}$ in $[h^{0,2}(B_R)]^k$, converges in the $[W^{s,2}(B_R)]^k$-norm and $Pu = f$ in B_R.

Proof. This follows from Corollary 2.4, (1.2) and the fact that

$$\Pi(S_P^{s,2}(B_R)) \tilde{\mathcal{G}} Tf = \lim_{\nu\to\infty} \tilde{\mathcal{G}}^{\nu} (\tilde{\mathcal{G}} Tf) = \lim_{\nu\to\infty} \tilde{\mathcal{G}}^{\nu} (\tilde{\mathcal{G}} Tf - \tilde{\mathcal{G}}^2 Tf) = 0. \quad \square$$

We emphasize that the coefficients $C_{\nu}^{(i)}$ in (2.6) do not depend on s and m. In the next section we discuss in detail the existence of $[W^{s,2}(B_R)]^k$-solutions of the equation $Pu = f$ and obtain a formula for its solutions with data in $[W^{m,2}(B_R)]^l$ $(m \ge 0)$.

§3. On the solvability of the system $Pu = f$ in B_R

In this section we obtain a criterion for the existence of $[W^{s,2}(B_R)]^k$-solutions the system

$$Pu = f \text{ in } B_R$$

and a formula for its $[W^{m+1,2}_{loc}(B_R)]^k$-solutions with the datum $f \in [W^{m,2}(B_R)]^l$ $(m \ge 0)$.

Because P is a system of partial differential operators with constant coefficients and injective principal symbol, it can be included into an elliptic Hilbert compatibility complex

$$0 \to [C^{\infty}(\mathbb{R}^n)]^k \xrightarrow{P} [C^{\infty}(\mathbb{R}^n)]^l \xrightarrow{P^1} [C^{\infty}(\mathbb{R}^n)]^N \xrightarrow{P^2} \dots . \quad (3.1)$$

with $P^\circ = P$. This means that P^1 is a differential operator with constant coefficients of order $p_1 \geq 1$,

$$P^{i+1} \circ P^i = 0$$

and that

$$\mathbb{C}^k \xrightarrow{\sigma(P)(\zeta)} \mathbb{C}^l \xrightarrow{\sigma_{P1}(P^1)(\zeta)} \mathbb{C}^N$$

is an exact sequence for every $\zeta \in \mathbb{R}^n \backslash \{0\}$.

One easily sees that the condition $P^1 f = 0$ is a necessary one for the solvability of the equation $Pu = f$.

Let us denote by $S^{m,2}_{P^1,P^*}(B_R)$ the following closed subspace of $[W^{m,2}(B_R)]^l$:

$$S^{m,2}_{P^1,P^*}(B_R) = \{g \in [W^{m,2}(B_R)]^l : \ P^1 g = 0, \ P^* g = 0 \ in B_R\}.$$

Lemma 3.1. *For every $m \in \mathbb{Z}_+$, the system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ is an orthogonal basis in the Hilbert space $S^{m,2}_{P^1,P^*}(B_R)$. Moreover there exist constants $\widetilde{C}_1(m,n)$, $\widetilde{C}_2(m,n) >$ 0 such that*

$$\widetilde{C}_1(m,n) \|Ph^{(i)}_\nu\|^2_{W^{m,2}(B_R)} \leq \nu^{2m} \|Ph^{(i)}_\nu\|^2_{L^2(B_R)} \leq \widetilde{C}_2(m,n) \|Ph^{(i)}_\nu\|^2_{W^{m,2}(B_R)}.$$

Proof. The orthogonality and the estimates follow from the Stokes' formula and Lemmata 1.1, 1.2 by direct calculations.

Since the compatibility complex (3.1) is elliptic, for every $g \in S^{m,2}_{P^1,P^*}(B_R)$, there exists $v \in [W^{m+1,2}_{loc}(B_R)]^k$, satisfying $Pv = g$ in B_R (see, for example, [AnNa]). In particular, for every $0 < r < R$, $v \in [h^{s,2}(B_r)]^k$,

$$v = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} c^{(i)}_\nu(v,r) h^{(i)}_\nu, \ g = Pv = \sum_{\nu=1}^{\infty} \sum_{i=1, \lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} c^{(i)}_\nu(v,r) Ph^{(i)}_\nu,$$

where the first series converges in the $[W^{s,2}(B_r)]^k$-norm (and hence, due to Stiltjes-Vitaly Theorem, uniformly together with all the derivatives on compact subsets of B_r).

Because of Lemma 1.4, the coefficients $c^{(i)}_\nu(v,r)$ do not depend on r. Moreover, due to the orthogonality the system $\{Ph^{(i)}_\nu\}_{\lambda^{(i)}_\nu \neq 1}$, $c^{(i)}_\nu(v,r)$ depend only on g and do not depend on v. Hence the statement of the lemma holds. $\square$

Now, for $m \geq 0$ ($m \notin \mathbb{Z}_+$) we provide the space $S^{m,2}_{P^1,P^*}(B_R)$ with the Hermitian form

$$(u,v)_{S^{m,2}_{P^1,P^*}(B_R)} = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} K^{(i)}_\nu(f) \overline{K^{(i)}_\nu(g)} \nu^{2m} \ (f, g \in S^{m,2}_{P^1,P^*}(B_R)),$$

where $K^{(i)}_\nu(f)$ are the Fourier coefficients of the vector-function f with respect to the orthonormal basis $\{Ph^{(i)}_\nu\}$ in $S^{0,2}_{P^1,P^*}(B_R)$.

The following Proposition follows from Lemma 3.1 as Proposition 1.5 follows from Lemma 1.4.

Proposition 3.2. *For every $m \geq 0$, the Hermitian form $(.,.)_{S^{m,2}_{P^1,P^*}(B_R)}$ is a scalar product in the space $S^{m,2}_{P^1,P^*}(B_R)$ defining the topology, equivalent to the original one. Moreover, the system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ is an orthogonal basis in $S^{m,2}_{P^1,P^*}(B_R)$ and there exist constants $\widetilde{C}_1(m,n)$, $\widetilde{C}_2(m,n) > 0$ such that*

$$\widetilde{C}_1(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(B_R)} \leq \nu^{2m}\|Ph^{(i)}_\nu\|^2_{L^2(B_R)} \leq \widetilde{C}_2(m,n)\|Ph^{(i)}_\nu\|^2_{W^{m,2}(B_R)}.$$

In the following corollary $K^{(i)}_\nu(f-PTf)$, are the Fourier coefficients of the vector $f-PTf$ with respect to the orthogonal system $\{Ph^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ in $[L^2(B_R)]^l$:

$$K^{(i)}_\nu(f-PTf) = \frac{((f-PTf),\, Ph^{(i)}_\nu)_{L^2(B_R)}}{\|Ph^{(i)}_\nu\|^2_{L^2(B_R)}}.$$

Corollary 3.3. *For every $f \in [W^{m,2}(B_R)]^l$ $(m \geq 0)$, with $P^1 f = 0$ in B_R the vector-function*

$$u = Tf + \sum_{\nu=1}^{\infty} \sum_{i=1,\lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} K^{(i)}_\nu(f-PTf)h^{(i)}_\nu,$$

where the series converges in $[W^{s,2}(B_r)]^k$-norm for every $0 < r < R$, satisfies $Pu = f$ in B_R.

Proof. Since $T : [W^{m,2}(B_R)]^l \to [W^{m+1,2}(B_R)]^k$, using Stokes' formula one easily checks that

$$P^*(f-PTf) = 0,\ P^1(f-PTf) = 0 \ in\ B_R.$$

Moreover, Proposition 3.2 implies that, for every $m \geq 0$,

$$\frac{((f-PTf),\, Ph^{(i)}_\nu)_{L^2(B_R)}}{\|Ph^{(i)}_\nu\|^2_{L^2(B_R)}} = \frac{((f-PTf),\, Ph^{(i)}_\nu)_{W^{m,2}(B_R)}}{\|Ph^{(i)}_\nu\|^2_{W^{m,2}(B_R)}}\ (Ph^{(i)}_\nu \neq 0).$$

Now, arguing as in the proof of Lemma 3.1, we obtain that the statement of the corollary holds. □

Theorem 3.4. *Let $m \geq 0$ and $0 \leq s \leq m+1$. Then the following conditions are equivalent:*

(1) *for every $f \in [W^{m,2}(B_R)]^l$, with $P^1 f = 0$ in B_R there exists $u \in [W^{s,2}(B_R)]^k$, satisfying $Pu = f$ in B_R;*

(2)
$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s} \left| \frac{C^{(i)}_\nu(\widetilde{G}Tf)}{1-\lambda^{(i)}_\nu} \right|^2 < \infty$$

for every $f \in [W^{m,2}(B_R)]^l$, with $P^1 f = 0$ in B_R.

(3)
$$\sum_{\nu=1}^{\infty} \sum_{i=1,\lambda^{(i)}_\nu \neq 1}^{dim S_k(\nu)} R^{2\nu}\nu^{2s} \left| K^{(i)}_\nu(f-PTf) \right|^2 < \infty$$

for every $f \in [W^{m,2}(B_R)]^l$, with $P^1 f = 0$ in B_R;

(4) *there exists a constant $C_1 > 0$ such that*

$$\max_{\lambda_\nu^{(i)} \neq 1} \frac{1}{1-\lambda_\nu^{(i)}} \leq C_1 \nu^{2-2s+2m} \text{ for every } \nu \geq 1,\ 1 \leq i \leq dim S_k(\nu);$$

(5) *there exists a constant $C_2 > 0$ such that*

$$\min_{Ph_\nu^{(i)} \neq 0} \|Ph_\nu^{(i)}\|^2_{L^2(B_1)} \geq C_2 \nu^{2s-2m} \text{ for every } \nu \geq 1,\ 1 \leq i \leq dim S_k(\nu).$$

Proof. The equivalence of (1), (2) and (3) follows from Proposition 1.5, Corollary 2.5 and Corollary 3.3 immediately.

Condition (5) implies (3) because of Lemma 1.3 and Proposition 1.5.

Further, Proposition 3.2 and Corollary 3.3 imply that the image $\overline{Im\, P}$ of the operator

$$P : [h^{s,2}(B_R)]^k \to S^{m,2}_{P^1,P^*}(B_R)$$

is closed. Then (1) yields that there exists a constant $C_0 > 0$ such that

$$\|u\|^2_{W^{s,2}(B_R)} \leq C_0 \|Pu\|^2_{W^{m,2}(B_R)}$$

for every $u \in \left(S_P^{s,2}(B_R)\right)^\perp$, with $(Pu) \in W^{m,2}(B_R)$, where $\left(S_P^{s,2}(B_R)\right)^\perp$ is the orthogonal complement of $S_P^{s,2}(B_R)$ in $[h^{s,2}(B_R)]^k$ (cf. [Hö]). In particular,

$$\frac{R^{2\nu}\nu^{2(s)}}{C_2(s,n)} \leq \|h_\nu^{(i)}\|^2_{W^{s,2}(B_R)} \leq C_0 \|Ph_\nu^{(i)}\|^2_{W^{m,2}(B_R)} \leq \frac{C_0\, R^{2\nu}\, \nu^{2m}}{\widetilde{C}_1(m,n)} \|Ph_\nu^{(i)}\|^2_{L^2(B_1)}$$

for every $h_\nu^{(i)}$ with $\lambda_\nu^{(i)} \neq 1$. Therefore (1) implies (5).

Finally, (1.2) implies that (4) and (5) are equivalent. □

Corollary 3.5. *One can find a finite number $a \geq -1$ (depending on the operator P) that, for every $f \in [W^{a+s,2}(B_R)]^l$ ($s \geq 0$, $a+s \geq 0$) satisfying $P^1 f = 0$ in B_R, there exists a $[W^{s,2}(B_R)]^k$-solution u to $Pu = f$ in B_R.*

Proof. It follows, for example, from Lemma 3.2 of [Sh1] and Proposition 1.5 that the system $\{h_\nu^{(i)}\}$ is a basis in the space $[h^\infty(\overline{B}_R)]^k$ of harmonic vector-functions in B_R belonging to $[C^\infty(\overline{B}_R)]^k$. Then, for every $u \in [h^\infty(\overline{B}_R)]^k$, the series

$$u = \sum_{\nu=0}^{\infty} \sum_{i=1}^{dim S_k(\nu)} C_\nu^{(i)}(u) h_\nu^{(i)}$$

converges in $[C^\infty(\overline{B}_R)]^k$, and the series

$$u_1 = \sum_{\nu=1}^{\infty} \sum_{i=1, Ph_\nu^{(i)} \neq 0}^{dim S_k(\nu)} C_\nu^{(i)}(u) h_\nu^{(i)}$$

converges in $[W^{s,2}(B_R)]^k$ for every $s \geq 0$. According to Sobolev Embedding Theorems, $u_1 \in [C^q(\overline{B}_R)]^k$ for every $q \geq 0$, i.e. $u_1 \in [C^\infty(\overline{B}_R)]^k$. Hence one easily conclude that the series u_1 converges in $[C^\infty(\overline{B}_R)]^k$.

It is known (see, for example, [AnNa]) that, for every $g \in [C^\infty(\overline{B}_R)]^l$ satisfying $P^1 g = 0$ in B_R, there exists a vector $v \in [C^\infty(\overline{B_R})]^k$ with $Pv = g$ in B_R. Therefore the operator

$$P_\infty : \left(C^\infty_P(\overline{B}_R)\right)^\perp \to \{g \in [C^\infty(\overline{B}_R)]^l : \, P^1 g = 0, \; P^* g = 0 \; in \; B_R\}$$

(where $\left(C^\infty_P(\overline{B}_R)\right)^\perp$ stands for the closure of the linear span of the system $\{h^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ in $[C^\infty(\overline{B}_R)]^k$) is injective, surjective and continuous. The Open Mapping Theorem for the Frechet spaces implies the inverse operator P^{-1}_∞ of P_∞ is continuos too.

Now, using Theorem 3.4, one easily concludes that there exists such a finite number $a \geq -1$ (depending on the operator P) that, for every $f \in [W^{a+s,2}(B_R)]^l$ ($s \geq 0$, $a+s \geq 0$) satisfying the integrability conditions, there exists a $[W^{s,2}(B_R)]^k$-solution u to $Pu = f$ in B_R, unless the operator P^{-1}_∞ is not continuos. $\square$

Corollary 3.5 can be proved using Ehrenpreis Fundamental Principle (see [Bj]).

Example 3.6. Let $n_1 \geq 1$, $n_2 \geq 1$, Q be $(l_1 \times k)$- matrix factorization of the Laplace operator in $\mathbb{R}^{n_1}_x$ and q be $(l_2 \times 1)$-matrix factorization of the Laplace operator in $\mathbb{R}^{n_2}_y$. Then the operator

$$P = \begin{pmatrix} Q_x \\ \frac{1}{k} q_y \otimes I_k \end{pmatrix}$$

is a matrix factorization of the Laplace operator in $\mathbb{R}^{n_1+n_2}$.

We assume that either the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ or the dimension of the vector space $S_q(\mathbb{R}^{n_2})$ is not finite. Then, Theorem 3.4 implies that, for every $m \geq 0$ and $s > m + 1/2$, the image $Im(P_{s,m})$ of the operator

$$P_{s,m} : \, [W^{s,2}(B_R)]^k \to [W^{m,2}(B_R)]^l$$

is not closed (cf. [Ke] and [NaSh] for the Cauchy-Riemann system).

Indeed, let the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ be not finite. We fix an eigenfunction $\widetilde{h}_1(y)$ of Green's operator $\mathcal{G}_q$ corresponding to a ball in $\mathbb{R}^{n_2}$ and to an eigenvalue $\widetilde{\lambda}_1 \neq 1$. Because the dimension of the vector space $S_Q(\mathbb{R}^{n_1})$ is not finite, for any number $N > 0$ there exists a number $\nu \geq N$ such that $Q\widetilde{h}_\nu(x) = 0$ in B_R, and therefore there exists a harmonic homogeneous polynomial $h_{\nu+1} = \widetilde{h}_1(y) h_\nu(x)$ in $\mathbb{R}^{n_1+n_2}$, with

$$\mathcal{G}_P h_{\nu+1} = \lambda_{\nu+1} h_{\nu+1}, \qquad \frac{1}{1 - \lambda_{\nu+1}} = \frac{n_1 + n_2 + 2\nu}{(n_1 + n_2)(1 - \widetilde{\lambda}_1)}.$$

Due to Theorem 3.4, $Im(P_{s,m})$ is not closed for every $s > m + 1/2$.

The proof for the case where the dimension of the vector space $S_q(\mathbb{R}^{n_2})$ is not finite, is similar. $\square$

Example 3.7. Let $n = 3$, $\mathbb{R}^3 = \mathbb{C}^1_z \times \mathbb{R}_x$, $l = 2$, $k = 1$, $P = \begin{pmatrix} 2\frac{\partial}{\partial \bar z} \\ \frac{\partial}{\partial x} \end{pmatrix}$. According to Example 3.6 we can not garantee the existence of $W^{s,2}(B_R)$-solutions of $Pu = f$ for all data in $[W^{m,2}(B_R)]^2$ satisfying the compatibility conditions if $s > m + 1/2$. However we can do it for $s = m + 1/2$.

Indeed, one easily checks that harmonic homogeneous polinomials of the type $h_\nu = \sum_{r+t+N=\nu} x^N h_{r,t}$ (where h_{rt} are the polinomials from Example 1.9) are dense in $h^{s,2}(B_R)$. Moreover, $\mathcal{G} h_\nu = \frac{1+N+2r}{1+2\nu} h_\nu$. Now Theorem 3.4 implies the desired statement. $\square$

As in [NaSh], we can easily apply the spectral decomposition of the operator G_s and Theorem 3.4 to the following P-Neumann Problem (cf. also [Ky], §§17–19).

Problem 3.8. *Let $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$ be a given vector, $m \geq 0$ and $0 \leq N \leq m+1$. It is requared to find $u \in [W^{N,2}(B_R)]^k$ such that*

$$\begin{cases} \Delta_n I_k u = 0 \textit{ in } B_R \\ \left(\sum_{i=1}^n P_i^* x_i\right) P u = \psi \textit{ on } B_R. \end{cases} \tag{3.2}$$

It follows from the Stokes' formula that the condition

$$\int_{|y|=R} (h_\nu^{(i)})^*(y)\psi(y) d\sigma(y) = 0 \textit{ for all } h_\nu^{(i)} \textit{ with } P h_\nu^{(i)} = 0 \tag{3.3}$$

is necessary, for Problem 3.8 to be solvable. Because the dimension of $S_P(\mathbb{R}^n)$ can be infinite, in general, Problem 3.8 is not an elliptic boundary value problem.

In the following corollary $\tilde\tau\psi$ stands for the integral

$$(\tilde\tau\psi)(x) = \frac{1}{R}\int_{|y|=R} \varphi_n(x-y)\psi(y) d\sigma(y)$$

and $C_\nu^{(i)}(\tilde\tau\psi)$ stands for the Fourier coefficients of the vector $\tilde\tau\psi$ with respect to the orthogonal basis $\{h_\nu^{(i)}\}$ in $[h^{0,2}(B_R)]^k$ (because $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$, using 2.3.2.5 in [ReSz], we conclude that $\tilde\tau\psi \in [h^{m+1,2}(B_R)]^k$).

Corollary 3.9. *If Problem 3.8 is solvable for every $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$, satisfying (3.3), then conditions in Theorem 3.4 hold with $s = N$, and*

$$u = \sum_{\nu=1}^{\infty} \sum_{i=1, P h_\nu^{(i)} \neq 0}^{dim S_k(\nu)} \frac{(n+2\nu-2)(n+2\nu)\, C_\nu^{(i)}(\tilde\tau\psi)}{R^{n+2\nu}\, \|P h_\nu^{(i)}\|^2_{L^2(B_1)}} h_\nu^{(i)}, \tag{3.4}$$

is its unique solution in $\left(S_P^{N,2}(B_R)\right)^\perp$. Back, if conditions in Theorem 3.4 hold with $s = (N+m+1)/2$, Problem 3.8 is solvable for every $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$, satisfying (3.3).

Proof. Since $\tau P u = \widetilde{\tau\psi}$ for a solution u of Problem 3.8, using Corollary 2.5 one easily concludes that formula (3.4) holds.

Let Problem 3.8 be solvable for every $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$, satisfying (3.3). Then, it is easy to see that (3.3) holds for $\psi = \left(\sum_{i=1}^n P_i^* x_i\right) f \in [W^{m-1/2,2}(\partial B_R)]^k$

with $f \in S^{m,2}_{P^1,P^*}(\partial B_R)$. Denoting by $u \in [h^{N,2}(B_R)]^k$ a solution of Problem 3.8 for such a vector ψ, we obtain that $\tau P u - \widetilde{\tau}\,(\sum_{i=1}^n P_i^* x_i)\, f = 0$. Because Lemma 1.1, Propositions 1.5 and 3.2, $f = Pu$, i.e. the conditions of Theorem 3.4 holds with $s = N$.

Back, because $\psi \in [W^{m-1/2,2}(\partial B_R)]^k$, there exists a function $v \in [h^{m,2}(B_R)]^k$ such that $v_{|\partial B_R} = \psi$. Condition (3.3) implies $v \in \left(S^{m,2}_P(B_R)\right)^\perp$. Hence we can decompose v with respect to the orthogonal basis $\{h^{(i)}_\nu\}_{Ph^{(i)}_\nu \neq 0}$ in this space. Denoting by $C^{(i)}_\nu(v)$ the corresponding Fourier coefficients, we set

$$\widetilde{u} = \sum_{\nu=1}^{\infty} \sum_{i=1, Ph^{(i)}_\nu \neq 0}^{dim S_k(\nu)} \frac{(n+2\nu)\, C^{(i)}_\nu(v)}{\|Ph^{(i)}_\nu\|^2_{L^2(B_1)}} h^{(i)}_\nu .$$

One easily calculates that $\widetilde{u} = u$ is a solution of Problem 3.8, if condition (5) of Theorem 3.4 holds with $s \geq (m+N+1)/2$. $\square$

Corollary 3.10. *For every $\psi \in [C^\infty(\partial B_R)]^k$, satisfying (3.3), there exists $u \in h^\infty[(\overline{B}_R)]^k$ satisfying (3.2).*

In the case where P is the gradient operator in $\mathbb{R}^n$, Problem 3.8 is the Neumann Problem and (3.4) is a classical formula for its solutions (cf., for example, [Vl], p. 426–428). For the Cauchy-Riemann system see [Ky], p. 181).

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