A HOMOTOPY OPERATOR FOR SPENCER COMPLEX IN C^∞-CASE

A. SHLAPUNOV AND N. TARKHANOV

Abstract. We show how the multiple application of the formal Cauchy-Kovalevskaya theorem leads to the main result of the formal theory of overdetermined systems of partial differential equations. Namely, any sufficiently regular system $Au = f$ with smooth coefficients on an open set $U \subset \mathbb{R}^n$ admits a solution in smooth sections of a bundle of formal power series, provided that f satisfies a compatibility condition in U.

Contents

Introduction

In this paper we deal with formal theory of overdetermined equations, although the case of determined equations is not excluded. By an overdetermined operator is meant any map $A : U \to V$ for which there exists a non-zero map $B : V \to W$ with the property that $BA = 0$. Then for the inhomogeneous equation $Au = f$ to be solvable it is necessary that $Bf = 0$. The formal theory of overdetermined equations consists in constructing a "smallest" map B with this property, i.e., any other map $C : V \to Z$ satisfying $CA = 0$ should act through B. This means, $C = QB$ for some map $Q : W \to Z$. If exists, such a map B is called compatibility operator for A.

Date: March 7, 2007.
2000 *Mathematics Subject Classification.* Primary 35N10; Secondary 58J10.
Key words and phrases. Overdetermined systems, formal complexes.

The existence of a compatibility operator for A is by no means obvious. If exists, B is not unique, for the composition $C = QB$ with any invertible map $Q : W \to Z$ is a compatibility operator for A. The proper algebraic framework for constructing a compatibility operator is given by the concept of a resolution of a module in homological algebra. While every module possesses a resolution by free modules, these latter need not be finitely generated, cf. [ML63]. Hence, the compatibility operator B guaranteed by homological algebra may be very crude. For linear differential equations $Au = f$ this approach gives satisfactory results only in two cases. The first of the two is the case of operators A with constant coefficients, where the question is settled by the Hilbert syzygy theorem. The second one is the case of operators A with real analytic coefficients, where the module is Noetherian, cf. [Bjo93].

In the case of differential operators A with smooth coefficients the formal theory was developed in the framework of differential topology, mostly due to the cohomological approach of Spencer, cf. [Spe69]. A central concept of this theory is the notion of sufficiently regular system of differential equations. Although the sufficient regularity property is verified within linear algebra, it is awkward. Each sufficiently regular system possesses a compatibility operator, which is a partial differential operator with smooth coefficients constructed in the framework of linear algebra, see [Spe69], [Pom78], [Tar95], etc.

Having granted a suitable compatibility operator for A, the question arises whether the condition $Bf = 0$ is not only necessary but also sufficient for the solvability of $Au = f$. The $\bar{\partial}$-problem in complex analysis shows that it is not the case in general. The solvability fails to take place even modulo finite dimensional subspaces of V unless the manifold is strictly pseudoconvex. However, for local operators A we can localise the problem, thus using the advantage of formal solvability.

By the formal solvability is actually meant the solvability in smooth sections of the infinite dimensional bundle of formal power series. Spencer and his school used for thus purpose the bundles of finite order jets, perhaps to not leave the standard setting of classical analysis, see [Spe69]. The bundle of formal power series has much in common with very popular nowadays deformation quantisation, cf. [Fed96]. In particular, the differential geometry of this bundle is essentially raised by a connection whose meaning is very transparent. Namely, this connection vanish if and only if the section of the formal series bundle comes from a section of the vector bundle in question. This crucial property readily yields that the connection commutes with every differential operator on the bundle of formal power series. This way the formal analysis of the inhomogeneous equation $Au = f$ readily leads to what is known as Spencer's first resolution of a sufficiently regular differential operator A.

Spencer's first resolution can be actually written for an inhomogeneous system $Au = f$ with arbitrary differential operator A, which is not necessarily sufficiently regular. Since the connection on the bundle of formal power series is flat, i.e., its curvature is zero, Spencer's first resolution is a complex. Its cohomology bears information on the solvability of $Au = f$, which is well understood in the case of sufficiently regular systems. If the system fails to be sufficiently regular, the complex in question lacks crucial regularity properties. Still the construction of a homotopy operator for Spencer's first resolution remains of central interest in the theory of overdetermined systems.

This work was intended as an attempt at motivating the role that is played by the homotopy operator for the existence theory, i.e., the local solvability of overdetermined systems.

1. The bundle of formal power series

Let $\mathcal{X}$ be a smooth manifold of dimension n. Given a smooth vector bundle E over $\mathcal{X}$ and an open set $U \subset \mathcal{X}$, we write $\mathcal{E}(U, E)$ for the space of all smooth sections of E over U.

Sections $u, v \in \mathcal{E}(U, E)$ are called equivalent at a point $p \in U$ if the difference $u - v$ vanishes up to the infinite order at p. The classes of equivalent sections of E at p form a vector space which is denoted by $J_p(E)$. If $E \cong U \times \mathbb{C}^k$ is trivial over U and $x = x(p)$ are local coordinates in U, then u and v are equivalent at p if and only if $\partial^\alpha(u - v) = 0$ at x for all multi-indices $\alpha \in \mathbb{N}_0^n$. Here, $\mathbb{N}_0$ stands for $N \cup \{0\}$, and

$$\partial^\alpha = \Big(\frac{\partial}{\partial x^1}\Big)^{\alpha_1} \dots \Big(\frac{\partial}{\partial x^n}\Big)^{\alpha_n}.$$

We can thus identify the equivalence class of a section $u \in \mathcal{E}(U, E)$ at p with the sequence

$$(u_\alpha(x))_{\alpha \in \mathbb{N}_0^n}$$

where $u_\alpha(x) = \partial^\alpha u(x)/\alpha!$. If $y = y(p)$ is another local chart about p, then the equivalence class of u at p is represented by $(u_\alpha(y))_{\alpha \in \mathbb{N}_0^n}$ where $u_\alpha(y) = \partial^\alpha u(y)/\alpha!$. By chain rule,

$$u_\alpha(y) = \sum_{|\beta| \le |\alpha|} t_{\alpha,\beta}(y) u_\beta(x) \tag{1.1}$$

where $t_{\alpha,\beta}(y)$ is an infinite lower triangle matrix whose entries are monomials of $\partial_y^\gamma x^1, \dots, \partial_y^\gamma x^n$ with $|\gamma| \le |\alpha| - |\beta| + 1$. Under the change of local frame in E the representation of the equivalence class of $u \in \mathcal{E}(U, E)$ at p transforms similarly to (1.1), with $t_{\alpha,\beta}(y)$ being derivatives of the transition matrix of E of order $|\alpha| - |\beta|$. We have thus given the structure of smooth vector bundle of infinite rank over $\mathcal{X}$ to the disjoint union

$$J(E) := \bigsqcup_{p \in \mathcal{X}} J_p(E).$$

The bundle $J(E)$ is said to be the bundle of formal power series with coefficients in E over $\mathcal{X}$. It just amounts to the bundle of infinite order jets of sections of the bundle E over $\mathcal{X}$, denoted by $J^\infty(E)$. The formal theory of [Spe69] makes use of the bundles $J^s(E)$ of jets of finite order $s \in \mathbb{N}_0$ rather than of $J^\infty(E)$. The bundle $J^0(E)$ is identified with E.

If U is a coordinate neighbourhood in $\mathcal{X}$, such that E is trivial over U, then any section u of $J^s(E)$ has representation

$$u(x, z) = \sum_{|\alpha| \le s} u_\alpha(x) z^\alpha \tag{1.2}$$

in U, where $x \in U$, $z \in \mathbb{C}^n$ and u_α are functions in U with values in $\mathbb{C}^\ell$. The variable z is invariantly interpreted as a vector of the complexified tangent space for $\mathcal{X}$ at the point x. By the very definition, u is smooth if all the u_α are smooth for some family $\{U\}$ covering $\mathcal{X}$.

For $r \leq s$, we denote by $\pi^{r,s}$ the natural projection $\pi^{r,s} : J^s(E) \to J^r(E)$. In local coordinates we get

$$\pi^{r,s}\Big(\sum_{|\alpha|\leq s} u_\alpha(x) z^\alpha\Big) = \Big(\sum_{|\alpha|\leq r} u_\alpha(x) z^\alpha\Big).$$

The map $j^s : \mathcal{E}(\mathcal{X}, E) \to \mathcal{E}(\mathcal{X}, J^s(E))$ that associates with a section $u \in \mathcal{E}(\mathcal{X}, E)$ its s-jet is a differential operator on $\mathcal{X}$. In a coordinate neighbourhood U in $\mathcal{X}$, over which E is trivial, it has the form

$$j^s u\,(x,z) = \sum_{|\alpha|\leq s} \frac{\partial^\alpha u(x)}{\alpha!} z^\alpha$$

for $(x,z) \in U \times \mathbb{C}^n$. Also in the case $s = \infty$ this operator is local, i.e., satisfies $\operatorname{supp} j^s u \subset \operatorname{supp} u$ for all $u \in \mathcal{E}(\mathcal{X}, E)$.

2. Compatibility operators

Given any smooth vector bundles E and F over $\mathcal{X}$, we write $\mathrm{Diff}^a(\mathcal{X}; E, F)$ for the space of all linear partial differential operators A of order $\leq a$ mapping sections of E to those of F. For any coordinate neighbourhood U with coordinates $x = (x^1, \ldots, x^n)$ in $\mathcal{X}$, such that both E and F are trivial over U, such an operator takes the form

$$A = \sum_{|\alpha|\leq a} A_\alpha(x) \partial^\alpha \tag{2.1}$$

where A_α are $(\ell \times k)$-matrices of smooth functions on U, k, ℓ being the ranks of E and F, respectively.

The operator A is said to be overdetermined if there exists a non-zero operator $B \in \mathrm{Diff}^b(\mathcal{X}; F, G)$ satisfying $B \circ A \equiv 0$.

Definition 2.1. An operator $B \in \mathrm{Diff}^b(\mathcal{X}; F, G)$ is called a compatibility operator for A if $B \circ A \equiv 0$ and for each operator $C \in \mathrm{Diff}^c(\mathcal{X}; F, H)$ with $C \circ A \equiv 0$ there is an operator $Q \in \mathrm{Diff}^q(\mathcal{X}; G, H)$, such that $C = Q \circ B$.

In order to treat the compatibility operator for A we invoke the theory of D-modules, see [Mal04], [Bjo93].

Denote by $\mathcal{E}(\mathcal{X})[D]$ the ring of scalar differential operators with smooth coefficients on $\mathcal{X}$. By the product of two operators in $\mathcal{E}(\mathcal{X})[D]$ is meant their composition, which is certainly non-commutative.

Write $\mathcal{E}(\mathcal{X})[D]_\ell$ for the free finitely generated left $\mathcal{E}(\mathcal{X})[D]$-module with the standard addition '+' and multiplication '$\cdot$' by elements of $\mathcal{E}(\mathcal{X})[D]$ from the left. More precisely, we interpret the elements of $\mathcal{E}(\mathcal{X})[D]_\ell$ as ℓ-rows with entries in $\mathcal{E}(\mathcal{X})[D]$ and set

$$a \cdot (a_1, \ldots, a_\ell) := (a \circ a_1, \ldots, a \circ a_\ell)$$

for all $a \in \mathcal{E}(\mathcal{X})[D]$. It is easy to see that

$$\begin{aligned} (b \cdot a) \cdot e &= (b \circ a) \circ e \\ &= b \circ (a \circ e) \\ &= b \cdot (a \cdot e) \end{aligned}$$

for all $e \in \mathcal{E}(\mathcal{X})[D]_\ell$ and $a, b \in \mathcal{E}(\mathcal{X})[D]$, and the distributivity axioms are obviously satisfied.

To construct a global compatibility operator for A it suffices to paste together local compatibility operators by using a partition of unity on $\mathcal{X}$. Hence, there is no loss of generality in assuming that both $E \cong \mathcal{X} \times \mathbb{C}^k$ and $F \cong \mathcal{X} \times \mathbb{C}^\ell$ are trivial. Then $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is given by an $(\ell \times k)$-matrix of scalar differential operators on $\mathcal{X}$. Thus, A induces a map of free finitely generated left $\mathcal{E}(\mathcal{X})[D]$-modules

$$\mathcal{E}(\mathcal{X})[D]_k \stackrel{h_1}{\leftarrow} \mathcal{E}(\mathcal{X})[D]_\ell,$$

where we define $h_1(e) = e \circ A$ for $e \in \mathcal{E}(\mathcal{X})[D]_\ell$.

Obviously, $M = \mathcal{E}(\mathcal{X})[D]_k / \operatorname{im} h_1$ bears the structure of a left $\mathcal{E}(\mathcal{X})[D]$-module. Indeed, given an equivalence class $[m] \in M$, we define $a \cdot [m] = [a \cdot m]$ for all $a \in \mathcal{E}(\mathcal{X})[D]$. Since

$$a \circ (m + e \circ A) = a \cdot m + (a \cdot e) \circ A$$

for all $e \in \mathcal{E}(\mathcal{X})[D]_\ell$, it follows that the definition is correct, i.e., it does not depend on the particular choice of representative $m \in [m]$..

It is well known that each module admits a free resolution, i.e., there exists a (possibly infinite) exact sequence

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots,$$

where $F_0, F_1, \dots$ are free right $\mathcal{E}(\mathcal{X})[D]$-modules. More precisely, M is the quotient F_0/H_0 of a free $\mathcal{E}(\mathcal{X})[D]$-module F_0 over a submodule H_0, H_0 is the quotient F_1/H_1 of a free $\mathcal{E}(\mathcal{X})[D]$-module F_1 over a submodule H_1, and so on, see for instance [ML63].

Of course, such a sequence is not unique. However, it is unique modulo homotopy equivalence. We note that $\operatorname{im} h_1 \cong \mathcal{E}(\mathcal{X})[D]_\ell / \ker h_1$. Since $H_1 = \ker h_1$ is a left $\mathcal{E}(\mathcal{X})[D]$-module, it is the quotient F_2/H_2 of a free $\mathcal{E}(\mathcal{X})[D]$-module F_2 over a submodule H_2, and so on. Denote by h_0 the canonical projection $\mathcal{E}(\mathcal{X})[D]_k \to M$. Then we arrive at a free resolution

$$0 \leftarrow M \stackrel{h_0}{\leftarrow} \mathcal{E}(\mathcal{X})[D]_k \stackrel{h_1}{\leftarrow} \mathcal{E}(\mathcal{X})[D]_\ell \stackrel{h_2}{\leftarrow} F_2 \leftarrow \dots \tag{2.2}$$

of $M = \mathcal{E}(\mathcal{X})[D]_k / \mathcal{E}(\mathcal{X})[D]_\ell \circ A$.

If F_2 is finitely generated, i.e., $F_2 = \mathcal{E}(\mathcal{X})[D]_m$, then we easily see that h_2 is induced by some differential operator $B \in \mathrm{Diff}^b(\mathcal{X}; F, G)$ via $h_2(e) = e \circ B$ for $e \in \mathcal{E}(\mathcal{X})[D]_m$. This readily gives a compatibility operator B for A, for the sequence (2.2) is exact. However, the ring $\mathcal{E}(\mathcal{X})[D]$ is not Noetherian and hence we can not guarantee in general that the module F_2 is finitely generated. If $\{e_i\}_{i \in I}$ is a basis for F_2, with some index set I, then we consider $h_2^{(i)} = h_2(e_i) \in \mathcal{E}(\mathcal{X})[D]_\ell$, thus obtaining

$$B = \{h_2^{(i)}\}_{i \in I}.$$

Quillen proved that if the operator A is 'sufficiently regular' then all the modules $F_2, F_3, \dots$ can be chosen to be finitely generated and, moreover, the resolution (2.2) is of finite length, see [Qui64].

Let us clarify this. For each operator $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ there is a bundle homomorphism $h(A) : J^a(E) \to F$, such that

$$A = h(A) \circ j^a.$$

In fact, in a coordinate neighbourhood U in $\mathcal{X}$ where A has representation (2.1) and $u \in J^a(E)$ has representation (1.2) we may set

$$h(A)u\,(x) = \sum_{|\alpha|\le a} A_\alpha(x)\,\alpha! u_\alpha(x)$$

for $x \in U$. It follows from the bundle structure of $J^a(E)$ that this actually defines a global bundle homomorphism $h(A)$ with the desired property.

For $s \ge a$, we consider a family of vector spaces

$$\mathcal{R}^s(p) = \ker\left(h(j^{s-a}A) : J^s_p(E) \to J^{s-a}_p(F)\right)$$

parametrised by the points $p \in \mathcal{X}$. It is easy to see that the restriction of $\pi^{r,s}$ to $\mathcal{R}^s(p)$ takes its values in $\mathcal{R}^r(p)$.

Definition 2.2. A differential operator $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is said to be sufficiently regular if:

1) The dimensions of vector spaces $\mathcal{R}^s(p)$ with $s \ge a$ do not depend on $p \in \mathcal{X}$.
2) For all $a \le r \le s$ the rank of the map $\pi^{r,s} : \mathcal{R}^s(p) \to \mathcal{R}^r(p)$ does not depend on $p \in \mathcal{X}$.

The condition 1) means that, for all $s \ge a$, the family

$$\mathcal{R}^s = \bigcup_{p\in\mathcal{X}} \mathcal{R}^s(p)$$

is a vector bundle over $\mathcal{X}$ (regularity).

The condition 2) is more subtle and says that $\pi^{r,s}(\mathcal{R}^s)$ is a vector subbundle of $J^r(E)$ for all $a \le r \le s$.

The concept of sufficient regularity plays a crucial role in Spencer's theory, cf. [Spe69]. Although being within linear algebra, the conditions 1) and 2) are too awkward to be efficiently verified in the general case. Nevertheless the regularity is very important for a compatibility operator to exist in the class of differential operators.

Example 2.3. Let $\mathcal{X} = \mathbb{R}$ and $a \in \mathcal{E}(\mathbb{R})$ satisfy $a(x) > 0$ for $x > 0$ and $a(x) = 0$ for $x \le 0$. Define $Au\,(x) = a(x)u(x)$ for $u \in \mathcal{E}(\mathbb{R})$. The operator A is differential of zero order and A is well known to be not sufficiently regular, see for instance Example 1.3.5 in [Tar95]. Moreover, A has no compatibility operator in the class of usual differential operators, i.e., the module F_2 in (2.2) can not be chosen to be finitely generated. Indeed, $\ker h_1$ consists of all differential operators with smooth coefficients vanishing for $x \ge 0$. Each compatibility operator for A in the class of differential operators has the form

$$Bf\,(x) = \sum_{j=0}^{b} B_j(x) f^{(j)}(x),$$

where B_j is an m-column of smooth functions on $\mathbb{R}$ satisfying $B_j(x) = 0$ for all $x \ge 0$. Obviously, we may restrict our attention to those B which have order zero, i.e.,

$$B = \begin{pmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{pmatrix}$$

where $b_i \in \mathcal{E}(\mathbb{R})$ vanish for $x \geq 0$. When specified in the ring $\mathcal{E}(\mathbb{R})[D]$, the family $\{b_i\}$ should be linearly independent over $\mathcal{E}(\mathbb{R})[D]$. We now observe that the zero order differential operator $Cf\,(x) = (b_1(x)/x)\,f(x)$ has smooth coefficients and satisfies $CA \equiv 0$. If there is a row $Q = (Q_1, \ldots, Q_m)$ of scalar differential operators with smooth coefficients satisfying $C = QB$, then $b_1(x) = xQB$ on $\mathbb{R}$. Since $b_1 = (1, 0, \ldots, 0)B$ and the family $\{b_i\}$ is linearly independent over $\mathcal{E}(\mathbb{R})[D]$, it follows $xQ_1 = 1$ and $Q_j = 0$ for $j > 1$. This is impossible if the coefficients of Q are smooth. Hence, A does not possess any compatibility differential operator. Take now a Hamel basis $\{e_i\}_{i\in I}$ for the vector space consisting of all functions $e \in \mathcal{E}(\mathbb{R})$ vanishing for $x \geq 0$. By the above, the $\mathcal{E}(\mathbb{R})[D]$-module F_2 just amounts to the free submodule of $\mathcal{E}(\mathbb{R})[D]$ generated by the system $\{e_i\}_{i\in I}$, and all the modules $F_3, F_4, \ldots$ are zero. We thus arrive at the free resolution of the $\mathcal{E}(\mathbb{R})[D]$-module $M := \mathcal{E}(\mathbb{R})[D]/\mathcal{E}(\mathbb{R})[D] \circ A$

$$0 \leftarrow M \overset{h_0}{\leftarrow} \mathcal{E}(\mathbb{R})[D] \overset{h_1}{\leftarrow} \mathcal{E}(\mathbb{R})[D] \overset{h_2}{\leftarrow} F_2 \leftarrow 0$$

where $h_1(e) = e \circ A$ and $h_2(e) = e$, the element e being thought of as that of $\mathcal{E}(\mathbb{R})[D]$.

It is worth pointing out that the compatibility operator $B = \{e_i\}_{i\in I}$ obtained in the framework of $\mathcal{E}(\mathbb{R})[D]$-modules does not give "proper" solvability conditions for the equation $Au = f$ in smooth functions. Indeed, the condition $Bf = 0$ yields only that $f(x) = 0$ for $x \leq 0$. However, for the existence of a smooth solution to the equation $Au = f$ it is necessary and sufficient that $f(x) = 0$ for $x \leq 0$ and the limit

$$\lim_{x\to 0+} \Big(\frac{d}{dx}\Big)^j \frac{f(x)}{a(x)}$$

exist for each $j = 0, 1, \ldots$.

With any short complex of differential operators

$$\mathcal{E}(\mathcal{X}, E) \overset{A}{\to} \mathcal{E}(\mathcal{X}, F) \overset{B}{\to} \mathcal{E}(\mathcal{X}, G) \tag{2.3}$$

we associate the family of complexes of linear maps of finite dimensional vector spaces

$$J_p^{s+b+a}(E) \overset{h(j^{s+b}A)}{\to} J_p^{s+b}(F) \overset{h(j^s B)}{\to} J_p^s(G) \tag{2.4}$$

parametrised by points $p \in \mathcal{X}$ of the underlying manifold and $s = 0, 1, \ldots$. The complex (2.3) is said to be formally exact if the complex (2.4) is exact for all $p \in \mathcal{X}$ and $s \in \mathbb{N}_0$. For a long complex on $\mathcal{X}$, the formal exactness means formal exactness of any short subcomplex.

Lemma 2.4. *Each formally exact complex of differential operators is a compatibility complex for the initial operator A.*

Proof. See for instance Proposition 1.3.11 in [Tar95]. □

It is worth pointing out that not any compatibility complex for a differential operator is formally exact.

If $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is a sufficiently regular differential operator, then the families of vector spaces $\mathcal{R}^s(p)$ parametrised by $p \in \mathcal{X}$ behave properly to be filtered as $\mathcal{R}^s(p) \hookrightarrow \mathcal{R}^r(p)$ for all $a \leq r \leq s$, the embeddings being of constant ranks. Under this condition a compatibility complex for A can be constructed purely within linear algebra.

Theorem 2.5. *For each sufficiently regular operator A on $\mathcal{X}$ one can construct in finitely many steps a formally exact complex $\{A^i\}_{i=0,1,\ldots,N}$ of differential operators on $\mathcal{X}$, such that $A^0 = A$.*

Proof. See [Qui64], [Gol67] or Theorem 3.3.9 in [Tar95]. □

3. Formal solutions to Hans Lewy's equation

Suppose that the system $Au = f$ has a sufficiently smooth solution u in a neighbourhood of a point $p \in U$. Write

$$\begin{aligned} u(x) &= \sum_{|\alpha|\le s} \frac{\partial^\alpha u(p)}{\alpha!}\,(x-p)^\alpha + o(|x-p|^s), \\ f(x) &= \sum_{|\alpha|\le s-a} \frac{\partial^\alpha f(p)}{\alpha!}\,(x-p)^\alpha + o(|x-p|^{s-a}) \end{aligned}$$

and

$$A = \sum_{|\beta|\le a} \Big(\sum_{|\alpha|\le s-a} \frac{\partial^\alpha A_\beta(p)}{\alpha!}\,(x-p)^\alpha + o(|x-p|^{s-a}) \Big) \partial^\beta$$

near p. On substituting these expansions into the equality $Au = f$ and equating the coefficients of the same powers $(x-p)^\alpha$ with $|\alpha| \le s-a$ on both sides of the equality we get

$$\sum_{|\alpha|\le s-a} \frac{\partial^\alpha (Au)(p)}{\alpha!}\,(x-p)^\alpha = \sum_{|\alpha|\le s-a} \frac{\partial^\alpha f(p)}{\alpha!}\,(x-p)^\alpha,$$

i.e., $j_p^{s-a}(Au) = j_p^{s-a} f$ for all $s \ge a$.

Since $j^{s-a}A = h(j^{s-a}A) \circ j^s$, where $h(j^{s-a}A)$ is the bundle homomorphism $J^s(E) \to J^{s-a}(F)$ defined above, we deduce that for the local solvability of $Au = f$ about a point p it is necessary that the system would possess a formal solution at p in the sense $h(j^{s-a}A)\, j_p^s u = j_p^{s-a} f$.

The extreme case $s = \infty$ corresponds to formal power series solutions at the point p. Homological algebra gives an efficient tool to examine this, for the ring of scalar differential operators whose coefficients are formal power series at p is Noetherian. Write $\mathcal{F}(p)[D]$ for this ring. As the coefficients of A are smooth, we may expand them as formal power series at p, thus specifying A as $(\ell \times k)$-matrix with entries in $\mathcal{F}(p)[D]$. This gives rise to a mapping of free finitely generated $\mathcal{F}(p)[D]$-modules

$$\mathcal{F}(p)[D]_k \overset{A}{\leftarrow} \mathcal{F}(p)[D]_\ell .$$

As the ring $\mathcal{F}(p)[D]$ is Noetherian, we get a finite free resolution

$$0 \leftarrow M \overset{h_0}{\leftarrow} \mathcal{F}(p)[D]_k \overset{h_1}{\leftarrow} \mathcal{F}(p)[D]_\ell \overset{h_2}{\leftarrow} \mathcal{F}(p)[D]_m \leftarrow \ldots \tag{3.1}$$

of $M = \mathcal{F}(p)[D]_k / \mathcal{F}(p)[D]^\ell \circ A$.

In this way we get an $(\ell \times m)$-matrix h_2 of scalar differential operators with coefficients being formal power series at p. It provides us with a compatibility operator B in the class of formal power series at p. It is clear that the coefficients of the operator B need not depend continuously on the point $p \in \mathcal{X}$. However, this can be the case even in very involved situations.

Example 3.1. Consider the operator A of Example 2.3. Obviously, for $x \leq 0$ each formal power series u at $p = x$ satisfies the equation $Au = f$, if f vanishes for $x \leq 0$. Since there are no divisors of zero in the ring of formal power series, we deduce that the solution to $Au = f$ is unique for $x > 0$. More precisely, given a formal power series

$$f(x, z) = \sum_{j=0}^{\infty} f_j(x) z^j$$

at $x > 0$, choose a smooth function g in a neighbourhood of x with the property that

$$\frac{1}{j!}\, g^{(j)}(x) = f_j(x)$$

for all $j = 0, 1, \ldots$. Then the formal power series

$$u(x, z) = \sum_{j=0}^{\infty} \frac{1}{j!} \Big(\frac{g}{a}\Big)^{(j)} (x)\, z^j$$

satisfies $Au = f$ at x. It is clear that the coefficients of u are independent of the particular choice of g. Hence, we can take as B the formal power series of any function $b \in \mathcal{E}(\mathbb{R})$ satisfying $b(x) > 0$ for $x < 0$ and $b(x) = 0$ for $x \geq 0$. Note that the coefficients of $u(x, z)$ need not depend smoothly on x, even if $f_j(x)$ do so. In order that there be a formal power series u with smooth coefficients satisfying $Au = f$ for x in a neighbourhood of 0, it is necessary and sufficient that each derivative

$$\Big(\frac{g}{a}\Big)^{(j)} (x)$$

would have finite limit when $x \to 0+$.

We now turn to the equation of Hans Lewy, see [Lew57]. Let $\mathcal{X} = \mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$, where $z = x^1 + \imath x^2$ and $t = x^3$. The operator of Hans Lewy is $A = \bar{\partial}_z + \imath z \partial_t$. This operator is known to be sufficiently regular, and its compatibility operator is $B = 0$. The inhomogeneous equation $Au = f$ is locally solvable for any real analytic function f. However, it fails in general to have any local solution if f is merely C^∞.

This shows that the ring $\mathcal{E}(\mathcal{X})[D]$ of scalar differential operators with smooth coefficients is not a good choice for constructing a compatibility operator in the category of smooth functions. It is conceivable that D-modules may not be the right tool here.

Fix any $x_0 = (z_0, t_0)$ in $\mathcal{X}$. When using the ring $\mathcal{F}(x_0)[D]$ we get $j_{x_0}(B) = 0$, for there are no divisors of zero in this ring.

Write

$$A = \bar{\partial}_z + \imath (z - z_0) \partial_t + \imath z_0 \partial_t,$$

then for any monomial $(z - z_0)^{\alpha_1} (\bar{z} - \bar{z}_0)^{\alpha_2} (t - t_0)^{\alpha_3}$ we obtain

$$\begin{array}{rcl} A\, (z - z_0)^{\alpha_1} (\bar{z} - \bar{z}_0)^{\alpha_2 + 1} (t - t_0)^{\alpha_3} & = & (\alpha_2 + 1)\, (z - z_0)^{\alpha_1} (\bar{z} - \bar{z}_0)^{\alpha_2} (t - t_0)^{\alpha_3} \\ & + & \imath \alpha_3\, (z - z_0)^{\alpha_1 + 1} (\bar{z} - \bar{z}_0)^{\alpha_2 + 1} (t - t_0)^{\alpha_3 - 1} \\ & + & \imath \alpha_3 z_0\, (z - z_0)^{\alpha_1} (\bar{z} - \bar{z}_0)^{\alpha_2 + 1} (t - t_0)^{\alpha_3 - 1}. \end{array} \tag{3.2}$$

If $\alpha_3 = 0$ then the last two terms on the right-hand side of (3.2) vanish, i.e., we have

$$A\,(z-z_0)^{\alpha_1}(\bar{z}-\bar{z}_0)^{\alpha_2+1} = (\alpha_2+1)\,(z-z_0)^{\alpha_1}(\bar{z}-\bar{z}_0)^{\alpha_2}.$$

Using (3.2) and induction in α_3 we immediately conclude that for every monomial $(z-z_0)^{\alpha_1}(\bar{z}-\bar{z}_0)^{\alpha_2}(t-t_0)^{\alpha_3}$ there exists a polynomial $\wp_\alpha(z-z_0, \bar{z}-\bar{z}_0, t-t_0)$ of degree $\alpha_1+\alpha_2+2\alpha_3+1$, whose coefficients are polynomials with respect to z_0 and rational functions with respect to α, such that

$$A\,\wp_\alpha(z-z_0, \bar{z}-\bar{z}_0, t-t_0) = (z-z_0)^{\alpha_1}(\bar{z}-\bar{z}_0)^{\alpha_2}(t-t_0)^{\alpha_3}.$$

On writing this in coordinates $x = (x^1, x^2, x^3)$ we see that for any formal power series

$$f(x,z) = \sum_{\alpha\in\mathbb{Z}_0^3} f_\alpha(x) z^\alpha$$

at $x \in \mathbb{R}^3$ there exists a formal power series

$$u(x,z) = \sum_{\alpha\in\mathbb{Z}_0^3} u_\alpha(x) z^\alpha$$

satisfying $Au = f$. Moreover, the coefficients $u_\alpha(x)$ can be chosen to smoothly depend on x, if the coefficients $f_\alpha(x)$ do so. The solution $u(x,z)$ is certainly not unique, because the jet of any holomorphic function of z independent of t satisfies $Au = 0$.

4. Connection on the bundle of formal power series

Let U be a coordinate neighbourhood in $\mathcal{X}$ over which the bundles E and F are trivial, and let $x = (x^1, \ldots, x^n)$ be coordinates in U.

Throughout the section we assume $s \in \mathbb{N}_0 \cup \{\infty\}$. In the case $s = \infty$ we set $s - a = \infty$ for any finite a.

Any section u of the bundle $J^s(E)$ has local representation

$$u(x,z) = \sum_{|\alpha|\le s} u_\alpha(x) z^\alpha$$

over U, where $(x,z) \in U \times \mathbb{C}^n$. By definition, u is smooth if all the coefficients u_α are smooth functions $U \to \mathbb{C}^k$ for some family $\{U\}$ covering $\mathcal{X}$ (then it is true for all families $\{U\}$).

Our next objective is to introduce first order differential operators $\mathfrak{d}^s$ on $\mathcal{X}$, which map sections of $J^s(E)$ to sections of $J^{s-1}(E)\otimes\Lambda^1$, where $\Lambda^q := \Lambda^q T^*\mathcal{X}$ stands for the bundle of exterior forms of degree $0 \le q \le n$ over $\mathcal{X}$. These operators play a key role in Spencer's theory and are actually induced by a connection $\mathfrak{d} := \mathfrak{d}^\infty$ on the bundle of formal power series with coefficients in E over $\mathcal{X}$. It will cause no confusion if we suppress in notation the dependence of $\mathfrak{d}^s$ on E, for the genuine bundle is always clear from context. On the other hand, $\mathfrak{d}^s$ are of universal character and hardly depend on E.

More precisely, we set

$$(\mathfrak{d}^s u)\,(x,z) = \sum_{|\alpha|\le s-1} \Big(du_\alpha(x) - \sum_{j=1}^{n} (\alpha_j+1) u_{\alpha+e_j}(x) dx^j \Big) z^\alpha \qquad (4.1)$$

in local coordinates, where e_j is the multi-index of length 1 in $\mathbb{N}_0^n$ whose k th component is 1, if $k = j$, and 0 otherwise. If s is finite, then (4.1) actually defines a global differential operator $\mathfrak{d}^s \in \mathrm{Diff}^1(\mathcal{X}; J^s(E), J^{s-1}(E) \otimes \Lambda^1)$, see [Spe69]. If $s = \infty$, this is no longer the case, for the bundle $J^\infty(E)$ is of infinite rank.

Lemma 4.1. *As defined above, $\mathfrak{d}$ is a connection on the bundle of formal power series with coefficients in E over $\mathcal{X}$.*

Proof. It suffices to show that $\mathfrak{d}$ fulfills the Leibniz formula $\mathfrak{d}(fu) = df\, u + f\mathfrak{d}u$ for all $u \in \mathcal{E}(\mathcal{X}, J(E))$ and $f \in \mathcal{E}(\mathcal{X})$. Since this formula is of local character, it suffices to verify it in each coordinate neighbourhood U in $\mathcal{X}$. This easily follows by using the explicit formula (4.1). □

A section $u \in \mathcal{E}(U, J^s(E))$ is said to be flat in U if $\mathfrak{d}^s u = 0$ in U. It is easily seen that $\mathfrak{d}^s u = 0$ in U if and only if

$$u_\alpha(x) = \frac{1}{\alpha!}\, \partial^\alpha u_0(x)$$

for all $x \in U$ and all $|\alpha| \leq s$. In other words, each flat section $u \in \mathcal{E}(U, J^s(E))$ stems from a smooth section $u_0 \in \mathcal{E}(U, E)$ by

$$\begin{aligned} u(x,z) &= j^s u_0\,(x,z) \\ &= \sum_{|\alpha| \leq s} \frac{\partial^\alpha u_0(x)}{\alpha!} z^\alpha \end{aligned}$$

for $(x,z) \in U \times \mathbb{C}^n$.

As usual, for each $0 \leq q \leq n$, the operator $\mathfrak{d}$ raises a sequence of first order differential operators $\mathfrak{d}^q$ on $\mathcal{X}$ mapping sections of $J(E) \otimes \Lambda^q$ to sections of $J(E) \otimes \Lambda^{q+1}$. The operators $\mathfrak{d}^q$ are uniquely determined by requiring the generalised Leibniz formula

$$\mathfrak{d}^q(fu) = df\, u + (-1)^q f\mathfrak{d}u \tag{4.2}$$

for all $u \in \mathcal{E}(\mathcal{X}, J(E))$ and $f \in \Omega^q(\mathcal{X})$.

Actually, for each pair $0 \leq q \leq n$ and s, there exists a first order differential operator $\mathfrak{d}^{s,q}$ on $\mathcal{X}$ which maps sections of $J^s(E) \otimes \Lambda^q$ to sections of $J^{s-1}(E) \otimes \Lambda^{q+1}$ and satisfies a suitably modified equation (4.2). The operator $\mathfrak{d}^{s,q}$ is defined locally in the following way. Each section $u \in \mathcal{E}(\mathcal{X}, J^s(E) \otimes \Lambda^q)$ has in U local representation

$$u(x,z) = {\sum_{\#I=q}}' \Big(\sum_{|\alpha| \leq s} u_{I,\alpha}(x) z^\alpha \Big) dx^I$$

for $(x,z) \in U \times \mathbb{C}^n$, where $u_{I,\alpha}$ are smooth functions on U with values in $\mathbb{C}^k$. The prime on the summation symbol means that the sum is over all increasing multi-indices $I = (i_1, \ldots, i_q)$ of integers $1 \leq i_1 < \ldots < i_q \leq n$, and $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_q}$. Then we set

$$(\mathfrak{d}^{s,q} u)\,(x,z) = {\sum_{\#I=q}}' \Big(\sum_{|\alpha| \leq s-1} \Big(du_{I,\alpha}(x) - \sum_{j=1}^n (\alpha_j + 1) u_{I,\alpha+e_j}(x) dx^j \Big) z^\alpha \Big) \wedge dx^I, \tag{4.3}$$

cf. (4.1).

Obviously, $\mathfrak{d}^{s,0} = \mathfrak{d}^s$. Similarly to the exterior derivative we will write $\mathfrak{d}^{s,q}$ simply $\mathfrak{d}^s$ also for $q > 0$, when no confusion can arise.

The elements of

$$\Omega^q(\mathcal{X}, J^s(E)) := \mathcal{E}(\mathcal{X}, J^s(E) \otimes \Lambda^q),$$

will be referred to as differential forms of degree q with coefficients in the bundle $J^s(E)$ on $\mathcal{X}$.

It is easy to check that

$$\begin{array}{rcll} (\mathfrak{d}^s u)(x,z) & = & \pi^{s-1,s}\, d\, u(x,z-x), & \text{if} \quad s<\infty, \\ (\mathfrak{d}^s u)(x,z) & = & d\, u(x,z-x), & \text{if} \quad s=\infty, \end{array}$$

for all $u \in \Omega^q(\mathcal{X}, J^s(E))$, the exterior derivative d acting in the variable x. Hence it follows that $\mathfrak{d}^{s-1}\mathfrak{d}^s = 0$ for finite s. For $s=\infty$ we get

$$\begin{array}{rcl} \mathfrak{d}^{q+1}\mathfrak{d}^q & = & \lim\limits_{s\to\infty} \mathfrak{d}^{s-1,q+1}\mathfrak{d}^{s,q} \\ & = & 0, \end{array}$$

meaning that the resulting infinite sum is formal. Assuming $s \geq n$ we thus arrive at the complex

$$0 \to \mathcal{E}(\mathcal{X},E) \xrightarrow{j^s} \mathcal{E}(\mathcal{X},J^s(E)) \xrightarrow{\mathfrak{d}^s} \Omega^1(\mathcal{X},J^{s-1}(E)) \xrightarrow{\mathfrak{d}^{s-1}} \dots \xrightarrow{\mathfrak{d}^{s-n+1}} \Omega^n(\mathcal{X},J^{s-n}(E)) \to 0. \tag{4.4}$$

Lemma 4.2. *Suppose that $s \geq n$. As defined above, complex (4.4) is exact at each step.*

Proof. The exactness at step 0 is obvious. Since flat jets stem from smooth sections of E, the exactness of (4.4) at step 1 is also clear. It remains to prove the exactness at steps ≥ 2.

Let U be a coordinate neighbourhood in $\mathcal{X}$ over which the bundle E is trivial. We next prove that the complex

$$\mathcal{E}(U,J^s(E)) \xrightarrow{\mathfrak{d}^s} \Omega^1(U,J^{s-1}(E)) \xrightarrow{\mathfrak{d}^{s-1}} \dots \xrightarrow{\mathfrak{d}^{s-n+1}} \Omega^n(U,J^{s-n}(E)) \to 0$$

is exact at each term $\Omega^q(U,J^{s-q}(E))$ for $q=1,\dots,n$.

For $r=0,1,\dots$, we denote by $\Sigma^r := \Sigma^r T^*\mathcal{X}$ the r-fold symmetric product of the cotangent bundle of $\mathcal{X}$. Any section $u \in \Omega^q(\mathcal{X}, E\otimes\Sigma^{r-q})$ has in U local representation

$$u(x,z) = \sum_{\#I=q}{}' \Big(\sum_{|\alpha|=r-q} u_{I,\alpha}(x) z^\alpha \Big) dx^I,$$

$u_{I,\alpha}$ being smooth functions on U with values in $\mathbb{C}^k$. These bundles naturally occur in the complex

$$0 \to \mathcal{E}(U,E\otimes\Sigma^r) \xrightarrow{\delta} \Omega^1(U,E\otimes\Sigma^{r-1}) \xrightarrow{\delta} \dots \xrightarrow{\delta} \Omega^n(U,E\otimes\Sigma^{r-n}) \to 0, \tag{4.5}$$

where

$$\delta u\,(x,z) = \sum_{\#I=q}{}' \Big(\sum_{|\alpha|=r-q-1} \Big(\sum_{j=1}^n (\alpha_j+1) u_{I,\alpha+e_j}(x) dx^j \Big) z^\alpha \Big) \wedge dx^I$$

for $u \in \Omega^q(U,E\otimes\Sigma^{r-q})$.

As is noted in [Spe69], δ actually acts as exterior derivative applied in $z \in \mathbb{R}^n$ to the form

$$\sum_{\#I=q}{}' \Big(\sum_{|\alpha|=r-q} u_{I,\alpha}(x) z^\alpha \Big) dz^I,$$

hence the complex (4.5) is exact.

We proceed to show that (4.4) is exact over U. Suppose $f \in \Omega^q(U, J^{s-q}(E))$ is of the form

$$f(x,z) = \sum_{\# I = q}{}' \Big(\sum_{|\alpha| \leq s-q} f_{I,\alpha}(x) z^{\alpha} \Big) dx^I,$$

where $1 \leq q \leq n$ and $f_{I,\alpha}$ are smooth functions on U with values in $\mathbb{C}^k$. For $0 \leq p \leq s-q$, we introduce

$$f_p(x,z) = \sum_{\# I = q}{}' \Big(\sum_{|\alpha| = p} f_{I,\alpha}(x) z^{\alpha} \Big) dx^I.$$

From the construction (4.3) of $\mathfrak{d}^s$ it follows immediately that $\mathfrak{d}^s f = 0$ if and only if $df_p - \delta f_{p+1} = 0$ in U for all $p = 0, 1, \ldots, s-q-1$. We are looking for a section $u \in \Omega^{q-1}(U, J^{s-q+1}(E))$ satisfying $\mathfrak{d}^s u = f$ in U. Obviously, this is equivalent to the system

$$du_p - \delta u_{p+1} = f_p,$$

for $0 \leq p \leq s-q$, in U, where

$$u_p(x,z) = \sum_{\# I = q-1}{}' \Big(\sum_{|\alpha| = p} u_{I,\alpha}(x) z^{\alpha} \Big) dx^I.$$

We may choose $\{u_{I,0}\}_{\# I = q-1}$ arbitrarily in $\mathcal{E}(U,E)$, for instance, $u_{I,0} \equiv 0$ in U. This determines u_0.

The above system is thus reduced to the system

$$\delta u_{p+1} = du_p - f_p \tag{4.6}$$

in U, for $p = 0, 1, \ldots, s-q$. As the complex (4.5) is exact, all we have to check is that δ applied to the right-hand side of (4.6) is equal to zero, i.e., $\delta\,(du_p - f_p) = 0$ in U, whenever $p = 0, 1, \ldots, s-q$.

Now we argue by induction. For $p = 0$ the equality holds automatically. Assume that $\delta\,(du_p - f_p) = 0$ is fulfilled for some $1 \leq p < s-q$. Then there is a form $u_{p+1} \in \Omega^{q-1}(U, E \otimes \Sigma^{p+1})$ satisfying $\delta u_{p+1} = du_p - f_p$ in U. Using the equality $\delta d + d\delta = 0$, we get

$$\begin{array}{rcl} \delta\,(du_{p+1} - f_{p+1}) & = & -d\delta u_{p+1} - \delta f_{p+1} \\ & = & -d\,(du_p - f_p) - \delta f_{p+1} \\ & = & df_p - \delta f_{p+1} \\ & = & 0, \end{array}$$

which completes the induction. We have thus established that the cohomology of (4.4) over U is zero.

It follows that the complex of sheaves associated to (4.4) is exact at each step. Hence, it gives a fine resolution of the sheaf $\mathcal{E}(\cdot, E)$ over $\mathcal{X}$ defined by $U \mapsto \mathcal{E}(U,E)$ for open sets U in $\mathcal{X}$. By the abstract de Rham theorem, the cohomology of (4.4) at $\Omega^q(\mathcal{X}, J^{s-q}(E))$ is isomorphic to $H^q(\mathcal{X}, \mathcal{E}(\cdot, E))$ for all $q = 1, \ldots, n$, see for instance Theorem 5.2.13 in [Tar95]. Since the sheaf $\mathcal{E}(\cdot, E)$ is fine, its global cohomology is zero at positive steps, see Corollary 5.2.3 *ibid.* This shows that the cohomology of (4.4) at steps ≥ 2 is zero, as desired. $\square$

For $s \geq a$, the differential operator $j^{s-a} \circ A \in \mathrm{Diff}^s(\mathcal{X}; E, J^{s-a}(F))$ is called the $(s-a)$ th prolongation of A. Prolongations of a differential operator A bring information on all possible differential consequences of the inhomogeneous system

$Au = f$. We have $j^{s-a}A = h(j^{s-a}A) \circ j^s$, where $h(j^{s-a}A)$ is a bundle homomorphism $J^s(E) \to J^{s-a}(F)$ uniquely determined by $j^{s-a}A$. Of course, $h(j^{s-a}A)$ acts on the sections of $J^s(E)$ by linear transformations in fibres $J^s_p(E)$ smoothly depending on $p \in \mathcal{X}$. In particular, it induces a homomorphism of $\mathcal{E}(\mathcal{X})$-modules $\mathcal{E}(\mathcal{X}, J^s(E)) \to \mathcal{E}(\mathcal{X}, J^{s-a}(E))$, for which we use the same notation. For $s = \infty$, the bundle $J^\infty(E)$ coincides with the bundle of formal power series with coefficients in E over $\mathcal{X}$. Thus, $h(j^\infty A)$ is a homomorphism of infinite rank vector bundles $J(E) \to J(F)$ over $\mathcal{X}$.

If $u \in \mathcal{E}(\mathcal{X}, J^s(E))$ has local representation $u(x,z) = \sum\limits_{|\alpha| \le s} u_\alpha(x) z^\alpha$ over U, then we get

$$h(j^{s-a}A)u\,(x,z) = \sum_{|\alpha| \le s-a} \Big(\sum_{\substack{|\beta| \le a \\ \beta \le \gamma \le \alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma} A_\beta(x)}{(\alpha+\beta-\gamma)!} \frac{\gamma!}{(\gamma-\beta)!}\, u_\gamma(x) \Big) z^\alpha \qquad (4.7)$$

for $(x,z) \in U \times \mathbb{C}^n$. This shows that the bundle homomorphism $h(j^\infty A)$ is given by an infinite matrix whose entries are supported below a secondary diagonal determined by the order of A.

Lemma 4.3. *For any $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$, $B \in \mathrm{Diff}^b(\mathcal{X}; F, G)$ and $s \ge a + b$, we have*

$$h(j^{s-a-b} \circ BA) = h(j^{s-a-b} \circ B)\, h(j^{s-a} \circ A).$$

Proof. For finite s the equality is well known, cf. Corollary 1.3.2 in [Tar95]. We restrict ourselves to $s = \infty$.

Let $s \in J_p(E)$, where $p \in \mathcal{X}$. Choose a section $u \in \mathcal{E}(\mathcal{X}, E)$, such that $j^\infty_p u = s$. By definition,

$$\begin{aligned} h(j^\infty \circ BA)s &= h(j^\infty \circ BA) j^\infty_p u \\ &= j^\infty_p (B(Au)). \end{aligned}$$

On the other hand,

$$\begin{aligned} j^\infty_p(B(Au)) &= h(j^\infty \circ B)\, j^\infty_p(Au) \\ &= h(j^\infty \circ B)\, h(j^\infty \circ A)\, j^\infty_p u \\ &= h(j^\infty \circ B)\, h(j^\infty \circ A)s, \end{aligned}$$

as desired. ∎

In particular, we have $\mathfrak{d}u = du - h(j^\infty d)u$ for all $u \in \Omega^q(\mathcal{X}, J(E))$, the exterior derivative acting in x.

Lemma 4.4. *For any integers s and q with $s - (q+1) \ge a$, the following diagram is commutative:*

$$\begin{array}{ccc} \Omega^q(\mathcal{X}, J^{s-q}(E)) & \xrightarrow{\mathfrak{d}^{s-q}} & \Omega^{q+1}(\mathcal{X}, J^{s-q-1}(E)) \\ \Big\downarrow {\scriptstyle h(j^{s-q-a}A)\otimes I} & & \Big\downarrow {\scriptstyle h(j^{s-q-1-a}A)\otimes I} \\ \Omega^q(\mathcal{X}, J^{s-q-a}(F)) & \xrightarrow{\mathfrak{d}^{s-q-a}} & \Omega^{q+1}(\mathcal{X}, J^{s-q-1-a}(F)) \end{array}$$

Proof. Since the mappings entering into the diagram are local, it suffices to prove the commutativity of the diagram in any coordinate neighbourhood U in $\mathcal{X}$ over which both E and F are trivial. Then we can use local representations of $\mathfrak{d}^s$ and $h(j^{s-a}A)$.

Let $u \in \Omega^q(\mathcal{X}, J^{s-q}(E))$. Then

$$\Big(h(j^{s-q-1-a}A) \otimes I\Big)\mathfrak{d}^{s-q} u\,(x,z) \;=\; \sum_{\#I=q}{}' \sum_{|\alpha|\le s-q-1-a} z^\alpha \times \sum_{\substack{|\beta|\le a\\ \beta\le\gamma\le\alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma}A_\beta(x)}{(\alpha+\beta-\gamma)!}\,\frac{\gamma!}{(\gamma-\beta)!}\Big(du_{I,\gamma}(x) - \sum_{j=1}^n (\gamma_j+1)u_{I,\gamma+e_j}(x)dx^j\Big)\wedge dx^I \tag{4.8}$$

for all $(x,z) \in U \times \mathbb{C}^n$. Similarly,

$$\mathfrak{d}^{s-q-a}\Big(h(j^{s-q-a}A)\otimes I\Big)u\,(x,z) \;=\; \sum_{\#I=q}{}' \sum_{|\alpha|\le s-q-1-a} z^\alpha \times \Big(d(h(j^{s-q-a}A)u_I)_\alpha(x) - \sum_{j=1}^n(\alpha_j+1)(h(j^{s-q-a}A)u_I)_{\alpha+e_j}(x)dx^j\Big)\wedge dx^I, \tag{4.9}$$

where $u_I(x,z) = \sum_{|\alpha|\le s-q} u_{I,\alpha}(x)z^\alpha$.

An easy computation shows that

$$(h(j^{s-q-a}A)u_I)_{\alpha+e_j}(x) = \sum_{\substack{|\beta|\le a\\ \beta-e_j\le\gamma\le\alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma}A_\beta(x)}{(\alpha+\beta-\gamma)!}\,\frac{(\gamma+e_j)!}{(\gamma+e_j-\beta)!}\,u_{I,\gamma+e_j}(x)$$

and

$$\begin{aligned} &d(h(j^{s-q-a}A)u_I)_\alpha(x)\\ &= \sum_{j=1}^n \sum_{\substack{|\beta|\le a\\ \beta-e_j\le\gamma\le\alpha+\beta-e_j}} \frac{\partial^{\alpha+\beta-\gamma}A_\beta(x)}{(\alpha+\beta-\gamma-e_j)!}\,\frac{(\gamma+e_j)!}{(\gamma+e_j-\beta)!}\,u_{I,\gamma+e_j}(x)dx^j\\ &+ \sum_{\substack{|\beta|\le a\\ \beta\le\gamma\le\alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma}A_\beta(x)}{(\alpha+\beta-\gamma)!}\,\frac{\gamma!}{(\gamma-\beta)!}\,du_{I,\gamma}(x). \end{aligned}$$

Using the fact that $\alpha_j+\beta_j = \gamma_j+1$, provided $\gamma = \alpha+\beta-e_j$, we immediately obtain

$$\begin{aligned} &d(h(j^{s-q-a}A)u_I)_\alpha(x) - \sum_{j=1}^n(\alpha_j+1)(h(j^{s-q-a}A)u_I)_{\alpha+e_j}(x)dx^j\\ &= \sum_{\substack{|\beta|\le a\\ \beta\le\gamma\le\alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma}A_\beta(x)}{(\alpha+\beta-\gamma)!}\,\frac{\gamma!}{(\gamma-\beta)!}\Big(du_{I,\gamma}(x) - \sum_{j=1}^n(\gamma_j+1)u_{I,\gamma+e_j}(x)dx^j\Big) \end{aligned}$$

for $x \in U$.

Hence it follows that the right-hand sides of (4.8) and (4.9) coincide, which establishes the lemma. □

5. Spencer's complex

Definition 5.1. A differential operator $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is said to be formally integrable if:

1) The operator A is sufficiently regular.
2) For each $p \in \mathcal{X}$, the map $\pi^{s-1,s} : \mathcal{R}^s(p) \to \mathcal{R}^{s-1}(p)$ is surjective whenever $s > a$.

Formal integrability of a differential operator A of order a means that, for every $s > a$, all differential consequences of order s of the system $Au = 0$ (i.e., consequences extracted by means of differentiations of any orders, equating mixed derivatives, and application of linear algebra for each $x \in \mathcal{X}$) may be actually obtained by way of differentiation of order no more than $s - a$, and application of linear algebra.

A sufficiently regular differential operator need not be formally integrable, see for instance Example 1.3.17 in [Tar95]. However, each sufficiently regular differential operator can be transformed to a formally integrable operator by using homotopy equivalence.

Two differential operators A_E of type $E^0 \to E^1$ and A_F of type $F^0 \to F^1$ on $\mathcal{X}$ are called equivalent if there exist differential operators M_i of type $F^i \to E^i$ and M_i^{-1} of type $E^i \to F^i$, for $i = 0, 1$, and differential operators h_1^E of type $E^1 \to E^0$ and h_1^F of type $F^1 \to F^0$, with the property that the following conditions are fulfilled:

$$
\begin{array}{lrcl}
1) & M_1 A_F - A_E M_0 & = & 0, \\
 & M_1^{-1} A_E - A_F M_0^{-1} & = & 0;
\end{array}
\qquad
\begin{array}{lrcl}
2) & M_0^{-1} M_0 & = & I - h_1^F A_F, \\
 & M_0 M_0^{-1} & = & I - h_1^E A_E,
\end{array}
$$

cf. the diagram

$$
\begin{array}{ccc}
\mathcal{E}(\mathcal{X}, F^0) & \underset{h_1^F}{\overset{A_F}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, F^1) \\
{\scriptstyle M_0}\big\downarrow\big\uparrow{\scriptstyle M_0^{-1}} & & {\scriptstyle M_1}\big\downarrow\big\uparrow{\scriptstyle M_1{}^{-1}} \\
\mathcal{E}(\mathcal{X}, E^0) & \underset{h_1^E}{\overset{A_E}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, E^1).
\end{array}
\qquad (5.1)
$$

The following lemma clarifies the role of the concept of homotopy equivalence in constructing a compatibility operator.

Lemma 5.2. *Let A_E and A_F be equivalent differential operators on $\mathcal{X}$. If for A_F there exists a compatibility complex then there exists a compatibility complex for A_E, too.*

Proof. See for instance Proposition 1.2.7 in [Tar95]. □

Our next objective is to explain how to transform any sufficiently regular differential operator to a formally integrable operator.

Lemma 5.3. *Let $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ be a sufficiently regular operator. Then there is a differential operator D, which can be constructed in finitely many steps, such that:*

1) *The operator DA is formally integrable.*

2) *A section $u \in \mathcal{E}(U, E)$ satisfies $DA\,u = 0$ in U if and only if $Au = 0$ in U.*
3) *The operators A and DA are equivalent.*

Proof. A formally integrable differential operator $\tilde{A}$ equivalent to A can be constructed from A by completely writing differential consequences of the equation $Au = 0$. The sufficient regularity of A guarantees that this procedure terminates in finitely many steps. The operator $\tilde{A}$ obtained this way has the form $\tilde{A} = (I \oplus D) \circ A$ for some differential operator D. Obviously, local solutions to the homogeneous equations $\tilde{A}u = 0$ and $Au = 0$ are the same. Moreover, a trivial verification shows that the operators $\tilde{A}$ and A are equivalent, see for instance Example 1.2.6 in [Tar95]. □

For $s \geq a$ and $p \in \mathcal{X}$, we denote by $\sigma^s(p)$ the kernel of the bundle homomorphism $\pi^{s-1,s} : \mathcal{R}^s(p) \to \mathcal{R}^{s-1}(p)$. If the operator A is sufficiently regular and $s > a$, then σ^s is a vector bundle over $\mathcal{X}$.

Often σ^s is called the symbolic bundle of the $(s-a)$th prolongation $j^{s-a} \circ A$ of A because it may be identified with the kernel of the bundle homomorphism $E \otimes \Sigma^s \to F \otimes \Sigma^{s-a}$ induced by $h(j^{s-a}A)$. One can easily verify that the restriction of the formal exterior derivative operator δ to $\sigma^{s-q} \otimes \Lambda^q$ maps to $\sigma^{s-q-1} \otimes \Lambda^{q+1}$ for any s and q with $s - q - 1 \geq a$. This gives rise to the complex of bundle homomorphisms

$$0 \to \sigma^s \xrightarrow{\delta} \sigma^{s-1} \otimes \Lambda^1 \xrightarrow{\delta} \sigma^{s-2} \otimes \Lambda^2 \xrightarrow{\delta} \dots \xrightarrow{\delta} \sigma^{s-n} \otimes \Lambda^n \to 0 \tag{5.2}$$

that is known as δ-complex of Spencer. It is not necessarily exact at all steps but is so at the steps 0 and 1.

One of possible definitions of involutive differential operators actually reads that a differential operator A is called involutive if the complex (5.2) is exact for all $s \geq a$.

Theorem 5.4. *For each sufficiently regular differential operator A on $\mathcal{X}$ there exists an integer $s_0 \geq a$, such that the complex (5.2) is exact for all $s \geq s_0$.*

Proof. See for instance 4.1 of [Pom78, Ch. 3]. □

For a vector bundle E over $\mathcal{X}$ it will be convenient to denote by $\mathcal{S}_E$ the sheaf of germs of differentiable sections of E. Thus, $\mathcal{S}_E(U) = \mathcal{E}(U, E)$ for each open set $U \subset \mathcal{X}$.

If $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is sufficiently regular then we have a suitable compatibility complex of sheaves

$$\mathcal{S}_E \xrightarrow{A} \mathcal{S}_F \xrightarrow{B} \mathcal{S}_G, \tag{5.3}$$

the pair $\{A, B\}$ being sometimes referred to as an overdetermined operator. The basic question of the existence theory of overdetermined systems consists of finding reasonable conditions on A which guarantee the exactness of (5.3). This means, for any point $p \in \mathcal{X}$ and any $f \in \mathcal{E}(U, F)$ satisfying $Bf = 0$ in a neighbourhood U of p, there should exist a possibly smaller neighbourhood $V \subset U$ of p and a section $u \in \mathcal{E}(V, E)$, such that $Au = f$ in V. The well-known examples of Lewy [Lew57] and Mizohata [Miz61] show that the sufficient regularity of A is not sufficient for the exactness of (5.3).

To study the cohomology of (5.3), Spencer introduced the following complex, see his survey [Spe69]. By Lemma 4.4, the operator $\mathfrak{d}^{s-q}$ maps $\Omega^q(U, \mathcal{R}^{s-q})$ to

$\Omega^{q+1}(U, \mathcal{R}^{s-q-1})$ for any open set $U \subset \mathcal{X}$, provided that $s-(q+1) \geq a$. Since $\mathfrak{d}^{s-q}$ has zero curvature, we arrive at the complex of sheaves

$$0 \to \mathcal{S}_{\ker A} \xrightarrow{j^s} \mathcal{S}_{\mathcal{R}^s} \xrightarrow{\mathfrak{d}^s} \mathcal{S}_{\mathcal{R}^{s-1}\otimes\Lambda^1} \xrightarrow{\mathfrak{d}^{s-1}} \dots \xrightarrow{\mathfrak{d}^{s-n+1}} \mathcal{S}_{\mathcal{R}^{s-n}\otimes\Lambda^n} \to 0 \qquad (5.4)$$

over $\mathcal{X}$, $\mathcal{S}_{\ker A}$ being the sheaf of germs of smooth solutions to $Au = 0$ over $\mathcal{X}$, cf. (4.4). This differential complex is called the first sequence of Spencer for the operator A.

Lemma 5.5. *The cohomology of (5.4) is independent of s, provided $s \geq s_0 + n - 1$, where s_0 is the number from Theorem 5.4.*

Proof. See [Spe69, p. 196]. □

We say that $s \in \mathbb{N}_0$ is in the stable range if it is large enough for the cohomology of (5.4) to be stable.

Theorem 5.6. *Let $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ be sufficiently regular and $\{A^i\}_{i=0,1,\dots}$ be a formally exact complex of differential operators on $\mathcal{X}$ with $A^0 = A$. Then the cohomologies of the complexes*

$$\begin{array}{l} 0 \to \mathcal{S}_{\ker A}(\mathcal{X}) \xrightarrow{j^s} \mathcal{E}(\mathcal{X}, \mathcal{R}^s) \xrightarrow{\mathfrak{d}^s} \Omega^1(\mathcal{X}, \mathcal{R}^{s-1}) \xrightarrow{\mathfrak{d}^{s-1}} \dots \xrightarrow{\mathfrak{d}^{s-n+1}} \Omega^n(\mathcal{X}, \mathcal{R}^{s-n}) \to 0, \\ 0 \to \mathcal{S}_{\ker A}(\mathcal{X}) \hookrightarrow \mathcal{E}(\mathcal{X}, E^0) \xrightarrow{A^0} \mathcal{E}(\mathcal{X}, E^1) \xrightarrow{A^1} \dots \xrightarrow{A^{n-1}} \mathcal{E}(\mathcal{X}, E^n) \to \dots \end{array}$$

are the same, if $s \in \mathbb{N}_0$ is in the stable range.

This result is due to Quillen and it is contained in his unpublished thesis [Qui64], cf. Theorem 10.1.

Proof. The relationship between the complexes in question is expressed by the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{S}_{\ker A}(X) & \xrightarrow{j^s} & \mathcal{E}(\mathcal{X}, \mathcal{R}^s) & \xrightarrow{\mathfrak{d}^s} & \Omega^1(\mathcal{X}, \mathcal{R}^{s-1}) & \xrightarrow{\mathfrak{d}^{s-1}} & \dots \\
 & & \big\downarrow{\scriptstyle\hookrightarrow} & & \big\downarrow{\scriptstyle\hookrightarrow} & & \big\downarrow{\scriptstyle\hookrightarrow} & & \\
0 & \to & \mathcal{E}(\mathcal{X}, E^0) & \xrightarrow{j^s} & \mathcal{E}(\mathcal{X}, J^s(E^0)) & \xrightarrow{\mathfrak{d}^s} & \Omega^1(\mathcal{X}, J^{s-1}(E^0)) & \xrightarrow{\mathfrak{d}^{s-1}} & \dots \\
 & & \big\downarrow{\scriptstyle A^0} & & \big\downarrow{\scriptstyle h(j^{s-a_0}A^0)} & & \big\downarrow{\scriptstyle h(j^{s-1-a_0}A^0)\otimes I} & & \\
0 & \to & \mathcal{E}(\mathcal{X}, E^1) & \xrightarrow{j^{s-a_0}} & \mathcal{E}(\mathcal{X}, J^{s-a_0}(E^1)) & \xrightarrow{\mathfrak{d}^{s-a_0}} & \Omega^1(\mathcal{X}, J^{s-1-a_0}(E^1)) & \xrightarrow{\mathfrak{d}^{s-1-a_0}} & \dots \\
 & & \big\downarrow{\scriptstyle A^1} & & \big\downarrow{\scriptstyle h(j^{s-a_0-a_1}A^1)} & & \big\downarrow{\scriptstyle h(j^{s-1-a_0-a_1}A^1)\otimes I} & & \\
 & & \dots & & \dots & & \dots & &
\end{array} \qquad (5.5)$$

where s is large. Since the complex $\{A^i\}_{i=0,1,\dots}$ is formally exact and the first Spencer sequence for the trivial operator is exact, the diagram is exact except possibly for the first row and first column. Thus by diagram chasing the cohomology of the first column is the same as the stable cohomology of the first Spencer sequence. □

The difficulty with the first Spencer sequence is that although the original equation can be elliptic, this sequence is almost never elliptic in the sense that its symbol sequence at every non-zero cotangent vector is exact. To remedy this difficulty, Spencer developed at the beginning of the 1960s a different method for constructing a resolution of the sheaf of solutions of an equations. We will not discuss here the so-called second sequence of Spencer which has better formal properties than the first one, see [Spe69].

6. Normalised operators

In this section we describe an explicit local construction of a compatibility operator for A. In this way we also obtain additional information on the local structure of sufficiently regular operators.

Definition 6.1. A differential operator $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is said to be normalised if:

1) The order of A is equal to 1, i.e., $a = 1$.
2) The operator A is formally integrable.
3) The operator A is involutive.
4) The principal symbol map $\sigma(A) : E \otimes T^*\mathcal{X} \to F$ is surjective.

The principal symbol map is defined by $\sigma(A)u = h(A)u$ for $u \in E \otimes T^*\mathcal{X}$, where $E \otimes T^*\mathcal{X}$ is identified within $J^1(E)$.

The first three conditions have already been discussed. The last condition 4) actually means that among the equations $Au = 0$ there are no purely algebraic equations for components $u_1, \ldots, u_k$ of u. If such equations occur, one can exclude them by canceling a number of the functions $u_1, \ldots, u_k$. Obviously, the transformed operator is equivalent to the initial one.

Theorem 6.2. *Each sufficiently regular operator $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ on $\mathcal{X}$ can be transformed in finitely many steps within the framework of differentiations and linear algebra in fibers of the bundles into an equivalent normalised differential operator.*

Proof. See Theorem 1.3.24 of [Tar95]. □

Two complexes of differential operators A_E^i of type $E^i \to E^{i+1}$ and A_F^i of type $F^i \to F^{i+1}$ on $\mathcal{X}$ are called homotopy equivalent if there exist differential operators M_i of type $F^i \to E^i$ and M_i^{-1} of type $E^i \to F^i$, for $i = 0, 1, \ldots$, and differential operators h_i^E of type $E^i \to E^{i-1}$ and h_i^F of type $F^i \to F^{i-1}$, for $i = 1, 2, \ldots$, such that:

$$1) \quad \begin{array}{rcl} M_{i+1}A_F^i - A_E^i M_i &=& 0, \\ M_{i+1}^{-1}A_E^i - A_F^i M_i^{-1} &=& 0; \end{array} \qquad 2) \quad \begin{array}{rcl} M_i^{-1}M_i &=& I - h_{i+1}^F A_F^i - A_F^{i-1} h_i^F, \\ M_i M_i^{-1} &=& I - h_{i+1}^E A_E^i - A_E^{i-1} h_i^E \end{array}$$

for $i = 0, 1, \ldots$, cf. the diagram

$$
\begin{array}{ccccc}
\mathcal{E}(\mathcal{X}, F^{i-1}) & \overset{A_F^{i-1}}{\underset{h_i^F}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, F^{i}) & \overset{A_F^{i}}{\underset{h_{i+1}^F}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, F^{i+1}) \\
{\scriptstyle M_{i-1}}\Big\downarrow\Big\uparrow{\scriptstyle M_{i-1}^{-1}} & & {\scriptstyle M_{i}}\Big\downarrow\Big\uparrow{\scriptstyle M_{i}^{-1}} & & {\scriptstyle M_{i+1}}\Big\downarrow\Big\uparrow{\scriptstyle M_{i+1}^{-1}} \\
\mathcal{E}(\mathcal{X}, E^{i-1}) & \overset{A_E^{i-1}}{\underset{h_i^E}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, E^{i}) & \overset{A_E^{i}}{\underset{h_{i+1}^E}{\rightleftarrows}} & \mathcal{E}(\mathcal{X}, E^{i+1}).
\end{array}
\tag{6.1}
$$

Lemma 6.3. *Let $\{A_E^i\}_{i=0,1,\ldots,N}$ and $\{A_F^i\}_{i=0,1,\ldots,N}$ be compatibility complexes for differential operators A_E and A_F, respectively, i.e., $A_E^0 = A_E$ and $A_F^0 = A_F$. Then, if the operators A_E and A_F are equivalent, the compatibility complexes are homotopy equivalent.*

Proof. This is actually a result of homological algebra. For a proof, see for instance Proposition 1.2.8 of [Tar95]. □

Let $A \in \mathrm{Diff}^1(\mathcal{X}; E, F)$ be a sufficiently regular first order operator. We choose a coordinate neighbourhood U in $\mathcal{X}$, over which the bundles E and F are trivial, with coordinates $x = (x^1, \ldots, x^n)$. The coordinate x^n is assumed to be chosen so that the derivative ∂_n appears in the local expression of A. Then one can decompose the fibers E and F over U into direct sums $\mathbb{C}^k = \mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$ and $\mathbb{C}^\ell = \mathbb{C}^{\ell_1} \oplus \mathbb{C}^{\ell_2}$ in such a way that $k_1 = \ell_2$ and, after a suitable isomorphism between $\mathbb{C}^{k_1}$ and $\mathbb{C}^{\ell_2}$, the operator A is written in the form

$$
Au = \begin{pmatrix} M^{(1)} & M^{(2)} \\ \partial_n + T^{(1)} & T^{(2)} \end{pmatrix} \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}, \tag{6.2}
$$

where the differential operators $M^{(1)}$, $M^{(2)}$ and $T^{(1)}$ do not contain the derivative ∂_n.

The following definition is of crucial importance in the local construction of a compatibility operator.

Definition 6.4. Commutativity relations are said to hold in (6.2) if, for some differential operator $S^{(1)}$ in U which does not contain differentiation with respect to x^n, we have

$$
\begin{array}{rcl}
M^{(1)} \left(\partial_n + T^{(1)}\right) & = & \left(\partial_n + S^{(1)}\right) M^{(1)}, \\
M^{(1)} T^{(2)} & = & \left(\partial_n + S^{(1)}\right) M^{(2)}
\end{array}
\tag{6.3}
$$

in U.

The importance of commutativity relations was first understood by Guillemin [Gui68].

Lemma 6.5. *Let commutativity relations hold in (6.2), and N be a compatibility operator for $(M^{(1)}, M^{(2)})$. Then*

$$
Bf = \begin{pmatrix} N & 0 \\ \partial_n + S^{(1)} & -M^{(1)} \end{pmatrix} \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}, \tag{6.4}
$$

is a compatibility operator for A in U, where $f = f^{(1)} \oplus f^{(2)}$ is a decomposition of $f \in \mathcal{E}(U)^\ell$ in accordance with the decomposition of F.

Proof. A trivial verification shows that $BA = 0$ in U. The proof of the fact that B is a "smallest" operator with this property is cumbersome. We refer the reader to [Sam81]. □

In order to possess a local representation (6.2) with commutativity relations fulfilled, the differential operator A should be of generic form.. Let us discuss this in more details. A covector $\xi_0 \in T^*_p\mathcal{X}$ is said to be quasiregular for A at a point $p \in \mathcal{X}$, if

$$\dim\ker\sigma(A)(p,\xi_0) = \min_{\xi\in T^*_p\mathcal{X}\setminus\{0\}} \dim\ker\sigma(A)(p,\xi).$$

For instance, each non-characteristic covector $\xi_0 \in T^*_p\mathcal{X}$ for a differential operator A is quasiregular.

Lemma 6.6. *Let A be an involutive formally integrable first order differential operator and $x = (x^1, \dots, x^n)$ a coordinate system in U, such that the covector dx^n is quasiregular for A at each point $p \in U$. Then commutativity relations hold in (6.2).*

Proof. See [Sam81]. □

Assuming the coefficients of the operator $S^{(1)}$ to be undetermined, we obtain from (6.3) a system of linear algebraic equations for the coefficients.

In this way we actually get an inductive procedure for constructing a compatibility operator.

Theorem 6.7. *Suppose that A is a normalised differential operator of type $E \to F$ on $\mathcal{X}$, and $U \subset \mathcal{X}$ is a coordinate neighbourhood over which the bundles E and F are trivial. Then, for an everywhere dense open set of coordinate systems $x = (x^1, \dots, x^n)$ in U:*

1) The bundles $E\,|_U$ and $F\,|_U$ may be decomposed into direct sums

$$\begin{array}{rcl} E\,|_U &=& E^{(1)} \oplus \ldots \oplus E^{(n+1)}, \\ F\,|_U &=& F^{(1)} \oplus \ldots \oplus F^{(n)} \end{array}$$

in such a way that $A = A_1 \oplus \ldots \oplus A_n$ in U, where

$$\begin{array}{rcl} A_j u = \partial_j \left(u^{(1)} \oplus \ldots \oplus u^{(j)}\right) &+& T^{(1)}_j(x, \partial_1, \dots, \partial_{j-1}) \left(u^{(1)} \oplus \ldots \oplus u^{(j)}\right) \\ &+& T^{(2)}_j(x, \partial_1, \dots, \partial_j) \left(u^{(j+1)} \oplus \ldots \oplus u^{(n+1)}\right); \end{array}$$

2) For every $1 \le j \le n$, the operator $A_1 \oplus \ldots \oplus A_j$ (which contains the variables $(x^{j+1}, \dots, x^n)$ as parameters) is normalised, and the covector dx^j is quasiregular for it at each point $p \in U$.

Proof. See [Sam81]. □

The representation of a normalised operator A, as in 1), 2) of Theorem 6.7, is called the normal form of (E.) Cartan.

Corollary 6.8. *For each normalised differential operator A on $\mathcal{X}$ one can construct in a finitely many steps a formally exact complex $\{A^i\}_{i=0,1,\dots,N}$ of normalised differential operators on $\mathcal{X}$, such that $A^0 = A$.*

Proof. See [Sam81]. □

7. Overdetermined systems of ODE's

Consider a first order system of ordinary differential operators on an open interval $\mathcal{X} \subset \mathbb{R}$,

$$\begin{array}{ccc} a_{1,1}\partial u_1 + \ldots + a_{1,k}\partial u_k & + & b_{1,1}u_1 + \ldots + b_{1,k}u_k = f_1, \\ \ldots & & \ldots \\ a_{\ell,1}\partial u_1 + \ldots + a_{\ell,k}\partial u_k & + & b_{\ell,1}u_1 + \ldots + b_{\ell,k}u_k = f_\ell, \end{array} \tag{7.1}$$

where $a_{i,j}$ and $b_{i,j}$ are $(\ell \times k)$-matrices of differentiable functions on $\mathcal{X}$, and f_i an ℓ-column of differentiable functions on $\mathcal{X}$.

Our goal is to find conditions on the right-hand side f_i both necessary and sufficient for the local solvability of (7.1). To this end, we pick a point $x_0 \in \mathcal{X}$ and look for a solution u_j to (7.1) in a neighbourhood of x_0. We now apply the Gauß algorithm to (7.1).

Keeping the coefficients at x_0 we first apply the Gauß algorithm to the variables $\partial u_1, \ldots, \partial u_k$, obtaining

$$\begin{array}{rrrrrr} a_{1,1}\partial u_1 + \ldots + & a_{1,m}\partial u_m + \ldots + & a_{1,k}\partial u_k + & b_{1,1}u_1 + \ldots + & b_{1,k}u_k & = f_1, \\ & \ldots & & \ldots & & \\ & a_{m,m}\partial u_m + \ldots + & a_{m,k}\partial u_k + & b_{m,1}u_1 + \ldots + & b_{m,k}u_k & = f_m, \\ & & & b_{m+1,1}u_1 + \ldots + & b_{m+1,k}u_k & = f_{m+1}, \\ & & & \ldots & & \\ & & & b_{\ell,1}u_1 + \ldots + & b_{\ell,k}u_k & = f_\ell \end{array} \tag{7.2}$$

with some new coefficients $a_{i,j}$ and $b_{i,j}$, the right-hand side f_i, and possibly reindexed unknown functions u_j. Note that m just amounts to the rank of the matrix $a_{i,j}$ at x_0, i.e..,

$$m = \operatorname{rank} \left(a_{i,j}(x_0) \right)_{\substack{i=1,\ldots,\ell \\ j=1,\ldots,k}} . \tag{7.3}$$

We now proceed by applying the Gauß algorithm to the variables $u_1, \ldots, u_k$ in the last $\ell - m$ equations (7.2). Since the Gauß algorithm includes possible reindexing of the variables, the triangle structure of the first m equations may be violated. However, the property (7.3) obviously survives under such transformations. We thus get

$$\begin{array}{rrrrrr} a_{1,1}\partial u_1 + \ldots + a_{1,k}\partial u_k + & b_{1,1}u_1 + \ldots + & b_{1,n}u_n + \ldots + & b_{1,k}u_k & = f_1, \\ \ldots & \ldots & \ldots & & \\ a_{m,1}\partial u_1 + \ldots + a_{m,k}\partial u_k + & b_{m,1}u_1 + \ldots + & b_{m,n}u_n + \ldots + & b_{m,k}u_k & = f_m, \\ & b_{m+1,1}u_1 + \ldots + & b_{m+1,n}u_n + \ldots + & b_{m+1,k}u_k & = f_{m+1}, \\ & & \ldots & & \\ & & b_{m+n,n}u_n + \ldots + & b_{m+n,k}u_k & = f_{m+n}, \\ & & & 0 & = f_{m+n+1}, \\ & & & \vdots & \\ & & & 0 & = f_\ell, \end{array} \tag{7.4}$$

with some new coefficients $a_{i,j}$ and $b_{i,j}$, the right-hand side f_i, possibly reindexed unknown functions u_j, and

$$n = \operatorname{rank} \left(b_{i,j}(x_0) \right)_{\substack{i=m+1,\ldots,\ell \\ j=1,\ldots,k}} . \tag{7.5}$$

Obviously, the ranks m and n do not depend on each other, for we can start with a system (7.2) of arbitrary form. Both m and n are $\leq k$ and $m + n \leq \ell$.

From (7.4) we readily deduce that for the local solvability of (7.1) near x_0 it is necessary that

$$\begin{array}{rcl} f_{m+n+1}(x_0) & = & 0, \\ & \vdots & \\ f_\ell(x_0) & = & 0. \end{array} \tag{7.6}$$

The case $n = 0$ is not excluded. In this case the conditions $f_{m+1} = \ldots = f_\ell = 0$ near x_0 are necessary and sufficient for the existence of a solution to (7.1) in a neighbourhood of x_0, provided that a non-degeneracy conditions for the coefficients is fulfilled. Indeed, it is sufficient to fix arbitrary $u_{m+1}, \ldots, u_k$ and to solve the initial problem for the first m equations in (7.2) with data at x_0, which is possible by the Peano theorem.

If $n \geq 1$, the task is to solve the subsystem of (7.4) that contains the unknown functions $u_1, \ldots, u_k$ only. This gives

$$\begin{array}{rl} u_1 = & f_{m+1}/b_{m+1,1} + c_{1,n+1}u_{n+1} + \ldots + c_{1,k}u_k, \\ & \ldots \\ u_n = & f_{m+n}/b_{m+n,n} + c_{n,n+1}u_{n+1} + \ldots + c_{n,k}u_k, \end{array} \tag{7.7}$$

and so the number of unknown functions is diminished. Substituting (7.7) into the first m equations of (7.4) yields

$$\begin{array}{l} a_{1,n+1}\partial u_{n+1} + \ldots + a_{1,k}\partial u_k + b_{1,n+1}u_{n+1} + \ldots + b_{1,k}u_k = a_1(x_0,\partial)f, \\ \qquad \ldots \qquad\qquad\qquad\qquad \ldots \\ a_{m,n+1}\partial u_{n+1} + \ldots + a_{m,k}\partial u_k + b_{m,n+1}u_{n+1} + \ldots + b_{m,k}u_k = a_m(x_0,\partial)f, \end{array} \tag{7.8}$$

where

$$a_i(x_0,\partial)f = f_i - \sum_{j=1}^{n} a_{i,j}\partial\left(f_{m+j}/b_{m+j,j}\right) - \sum_{j=1}^{n} b_{i,j}\left(f_{m+j}/b_{m+j,j}\right)$$

for $i = 1, \ldots, m$.

The system (7.8) is actually of the same form as (7.1), but the number of unknown functions in (7.8) is n less than that in (7.1). Moreover, the right-hand side of (7.8) contains the derivatives of $f_1, \ldots, f_{m+n}$. Hence, we can apply the Gauß algorithm once again, thus obtaining necessary conditions for solvability of (7.8) in the form

$$\begin{array}{rcl} a_{o+p+1}f\,(x_0) & = & 0, \\ & \vdots & \\ a_m f\,(x_0) & = & 0 \end{array} \tag{7.9}$$

along with a new system of the form (7.1) containing a smaller number of unknown functions.

This process terminates giving conditions on the right-hand side f of (7.1) which are necessary and sufficient for the solvability of this system in a neighbourhood of $x_0 \in \mathcal{X}$. By (7.6) and (7.9), they are of the form

$$\begin{array}{rcl} A_0 f & = & 0, \\ A_1 f & = & 0, \\ & \vdots & \\ A_Q f & = & 0, \end{array} \tag{7.10}$$

where A_i is a matrix of linear differential operator of order i near x_0.. The construction shows that A_i contains $m_{i-1} - (m_i + n_i)$ rows and ℓ columns, with $m_{-1} = \ell$. Thus, (7.10) contains $\ell - n_0 - \ldots - n_Q - m_Q$ equations, which suggests that the compatibility operator for this system is zero. Note that the order of (7.10) does not exceed $\ell - 2$.

It remains to make explicit the non-degeneracy condition for the coefficients of (7.1) which is used in the construction. We started by applying the Gauß algorithm to the matrix $(a_{i,j})$ at x_0 obtaining m linearly independent rows. Since the coefficients $a_{i,j}$ are continuous functions, the rang of the matrix is a lower semicontinuous function. Hence there is a neighbourhood U of x_0, such that $\mathrm{rank}(a_{i,j}(x)) \geq \mathrm{rank}(a_{i,j}(x_0))$ for all $x \in U$. If there is a point $x \in U$, such that $\mathrm{rank}(a_{i,j}(x)) > \mathrm{rank}(a_{i,j}(x_0))$, then the Gauß algorithm at x gives more than m linearly independent rows. However, these destroyed at the point x_0, thus resulting in singularities of the resolution operator. To avoid such a situation which should require special study we assume that the rang of $(a_{i,j})$ is constant in a neighbourhood of x_0, i.e.,

$$m = \mathrm{rank}\,(a_{i,j}(x))_{\substack{i=1,\ldots,\ell \\ j=1,\ldots,k}} . \tag{7.11}$$

for all $x \in U$, cf. (7.3).

The same remains true concerning the Gauß algorithm applied to the matrix $(b_{i,j})_{\substack{i=m+1,\ldots,\ell \\ j=1,\ldots,k}}$. We require

$$n = \mathrm{rank}\,(b_{i,j}(x))_{\substack{i=m+1,\ldots,\ell \\ j=1,\ldots,k}} \tag{7.12}$$

for all $x \in U$, otherwise we don't get any regular resolution operator on all of U.

The question arises whether (7.12) can be formulated in a more invariant way which is independent of the splitting of $(b_{i,j})$ caused by the transformation of $(a_{i,j})$. The answer seems to be negative, i.e., in these terms the non-degeneracy condition cannot be improved.

The same reasoning applies to (7.8), where the matrix $(a_{i,j})$ is constructed from the genuine matrices $(a_{i,j})$ and $(b_{i,j})$ of (7.1) by linear algebra. On the other hand, the matrix $(b_{i,j})$ in (7.8) is constructed not only from the elements of matrices $(a_{i,j})$ and $(b_{i,j})$ in (7.1), but also from their derivatives. The matrix $(b_{i,j})$ occurring this way at the last step is constructed from the derivatives of the genuine matrices $(a_{i,j})$ and $(b_{i,j})$ up to at most order $l - 1$.

Summarising, we conclude that the non-degeneracy condition in question for the coefficients of (7.1) consists of constant rank assumptions for some matrices explicitly constructed from the coefficients of the system (7.1) and their derivatives up to order $l - 1$.

8. A formal Cauchy-Kovalevskaya theorem

In this section we discuss a version of the Cauchy-Kovalevskaya theorem in the class of smooth sections of jet bundles over $\mathcal{X}$. For this purpose, given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we set $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$. This enables us to write the components of jets as $u_\alpha = u_{\alpha',\alpha_n}$.

Theorem 8.1. *Suppose* $\operatorname{rank} E = \operatorname{rank} F$, $A_{(0,a)} = I$ *and* $s \in \mathbb{N}_0 \cup \{\infty\}$ *satisfies* $s \geq a$. *Then, given any*

$$\begin{array}{rcl} f & \in & \mathcal{E}(U, J^{s-a}(F)), \\ u^{(j)} & \in & \mathcal{E}(U, J^{s-j}(E)), \quad j = 0, 1, \ldots, a-1, \end{array}$$

there exists a unique $u \in \mathcal{E}(U, J^s(E))$ *satisfying*

$$\begin{array}{rcl} h(j^{s-a}A)u\,(x,z) & = & f(x,z), \\ h(j^{s-j}\partial_n^j)u\,(x,(z',0)) & = & u^{(j)}(x,(z',0)), \quad j = 0, 1, \ldots, a-1, \end{array} \tag{8.1}$$

for all $(x,z) \in U \times \mathbb{C}^n$.

Proof. Fix $x \in U$. Using (4.7) we conclude that (8.1) is equivalent to the system of linear algebraic equations

$$\begin{array}{rcl} A_{(0,a)}(x)(\alpha_n + a)u_{\alpha+ae_n}(x) + \displaystyle\sum_{\substack{|\beta| \leq a \\ \beta \neq ae_n \\ \beta \leq \gamma \leq \alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma} A_\beta(x)}{(\alpha+\beta-\gamma)!} \frac{\gamma!}{(\gamma-\beta)!}\, u_\gamma(x) & = & f_\alpha(x), \\ u_{(\alpha',j)}(x) & = & u^{(j)}_{(\alpha',0)}(x), \end{array} \tag{8.2}$$

for $|\alpha| \leq s-a$ and for $j = 0, 1, \ldots, a-1$ and $|\alpha'| \leq s-j$.

We now argue by induction in $\alpha_n \in \mathbb{N}_0$. Indeed, the second part of equations in (8.2) implies readily that the coefficients $u_{(\alpha',j)}$ are uniquely determined for all $j = 0, 1, \ldots, a-1$ and $|\alpha'| \leq s-j$. By the very setting, these coefficients belong to $\mathcal{E}(E, U)$.

Let r be an integer with $a \leq r < s$. Suppose that all the coefficients $u_{(\alpha',j)}$ with $0 \leq j \leq r$ and $|\alpha'| \leq s-j$ are uniquely defined and belong to $\mathcal{E}(U, E)$. Then the first equations in (8.2) implies that

$$u_{(\alpha',r+1)}(x) = \frac{1}{r+1}\Big(f_{(\alpha',r+1-a)}(x) - \sum_{\substack{|\beta| \leq a \\ \beta \neq ae_n \\ \beta \leq \gamma \leq (\alpha',r+1-a)+\beta}} \frac{\partial^{\alpha+\beta-\gamma} A_\beta(x)}{(\alpha+\beta-\gamma)!} \frac{\gamma!}{(\gamma-\beta)!}\, u_\gamma(x)\Big) \tag{8.3}$$

for all $\alpha_n = r+1-a$ and $|\alpha'| \leq s-(r+1-a)$.

It is clear that $\gamma_n \leq r+1-a+\beta_n \leq r$ on the right-hand side of (8.3), i.e., all the coefficients u_γ are already uniquely determined and belong to $\mathcal{E}(U, E)$ by assumption. Therefore, the coefficients $u_{(\alpha',r+1)}$ with $|\alpha'| \leq s-(r+1)$ are uniquely defined, too, and belong to $\mathcal{E}(U, E)$.

Thus, we have proved that there exists a unique $u \in \mathcal{E}(U, J^s(E))$ satisfying (8.2) for all $x \in U$, as desired. □

For an increasing multi-index $J = (j_1, \ldots, j_k)$ with $1 \leq j_1 < \ldots < j_m \leq n$, we choose a group of variables $x^{(J)} = (x^{j_1}, \ldots, x^{j_m})$. Write $\mathfrak{d}^s_{x^{(J)}}$ for the "connection" acting in $x^{(J)}$, i.e.,

$$\big(\mathfrak{d}^s_{x^{(J)}} u\big)(x,z) = \sum_{|\alpha| \leq s-1} \Big(\sum_{j \in J} \Big(\partial_{x^j} u_\alpha(x) - (\alpha_j+1) u_{\alpha+e_j}(x)\Big) dx^j\Big) z^\alpha$$

cf. (4.1).

Lemma 8.2. *Under the hypothesis of Theorem 8.1, if moreover* $n \notin J$ *and*

$$\begin{array}{rcl} \mathfrak{d}^{s-a}_{x^{(J)}} f & = & 0, \\ \mathfrak{d}^{s-j}_{x^{(J)}} u^{(j)} & = & 0, \quad j = 0, 1, \ldots, a-1, \end{array}$$

in U, then $\mathfrak{d}^s_{x(J)} u = 0$ in U.

Proof. Since $n \notin J$, Lemma 4.4 yields

$$\begin{aligned}\Big(h(j^{s-1-a}A) \otimes I\Big)\mathfrak{d}^s_{x(J)} u\,(x,z) &= \mathfrak{d}^{s-a}_{x(J)} h(j^{s-a}A)u\,(x,z)\\ &= \mathfrak{d}^{s-a}_{x(J)} f\,(x,z)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Big(h(j^{s-1-j}\partial_n^j) \otimes I\Big)\mathfrak{d}^s_{x(J)} u\,(x,(z',0)) &= \mathfrak{d}^{s-j}_{x(J)} h(j^{s-j}\partial_n^j)u\,(x,(z',0))\\ &= \mathfrak{d}^{s-j}_{x(J)} u^{(j)}\,(x,(z',0))\\ &= 0\end{aligned}$$

for all $j = 0, 1, \ldots, a-1$. Using Theorem 8.1 we deduce that $\mathfrak{d}^s_{x(J)} u = 0$ in U, as desired. □

As defined above, the actions of A and $h(j^\infty A)u$ on sections of the formal series bundle $J(E) \cong J^\infty(E)$ coincide. Hence we will write $h(j^\infty A)$ simply A when no confusion can arise.

Lemma 8.3. *Let $\ell \geq k$ and $\operatorname{rank} A_{(0,a)}(x) = k$ for all $x \in U$. If $u \in \mathcal{E}(U, J^\infty(E))$ satisfies $Au = 0$ and $u_{(\alpha',j)} = 0$ for all $\alpha' \in \mathbb{N}_0^{n-1}$ and $0 \leq j \leq a-1$, then $u = 0$.*

Proof. From $h(j^\infty A)u = 0$ we conclude that

$$\sum_{\substack{|\beta| \leq a \\ \beta \leq \gamma \leq \alpha+\beta}} \frac{\partial^{\alpha+\beta-\gamma} A_\beta(x)}{(\alpha+\beta-\gamma)!}\, \frac{\gamma!}{(\gamma-\beta)!}\, u_\gamma(x) = 0 \tag{8.4}$$

in U for all $\alpha \in \mathbb{N}_0^n$.

We argue by induction with respect to $\alpha_n \in \mathbb{N}_0$. Setting $\alpha_n = 0$ in (8.4) yields $\beta \leq \gamma \leq (\alpha',0)+\beta$ whence $\gamma_n = \beta_n$. Since $|\beta| \leq a$, we get $\gamma_n \leq a$. By assumption, $u_{(\gamma',j)} = 0$ for all γ' and $0 \leq j \leq a-1$. Hence it follows for all multi-indices $\alpha' \in \mathbb{N}_0^{n-1}$ that

$$\sum_{\gamma' \leq \alpha'} \frac{\partial^{(\alpha'-\gamma',0)} A_{(0,a)}(x)}{(\alpha'-\gamma')!}\, a!\, u_{(\gamma',a)}(x) = 0$$

in U.

Substituting $\alpha' = 0$ into this equality gives $A_{(0,a)}(x)u_{(0,a)}(x) = 0$ at each point $x \in U$. Since the rank of $A_{(0,a)}$ is equal to k in U, we conclude that $u_{(0,a)}$ vanishes in U. Substituting $\alpha' = e'_j$ for $1 \leq j \leq n-1$ yields $A_{(0,a)}(x)u_{(\alpha',a)}(x) = 0$ for all $x \in U$, and so $u_{(\alpha',a)} = 0$ in U for all $\alpha' \in \mathbb{N}_0^{n-1}$ with $|\alpha'| = 1$, and so on. We can now proceed in this manner obtaining $u_{(\alpha',a)} = 0$ in U for all multi-indices $\alpha' \in \mathbb{N}_0^{n-1}$.

If now $u_{(\gamma',j)} = 0$ for all γ' and $0 \leq j \leq s$, where $s \geq a$, we apply the same reasoning again, with $\alpha_n = 0$ replaced by $\alpha_n = s-(a-1)$, to obtain $u_{(\alpha',s+1)} = 0$ in U for all multi-indices $\alpha' \in \mathbb{N}_0^{n-1}$.

We have thus proved that $u_\alpha = 0$ for all multi-indices $\alpha \in \mathbb{N}_0^n$, i.e., $u = 0$ in U, as desired.

□

9. Cohomology of formal power series

Throughout this section we assume that $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ is a sufficiently regular operator on $\mathcal{X}$.

Lemma 9.1. *Let A_E and A_F be equivalent differential operators of type $E^0 \to E^1$ and $F^0 \to F^1$ on $\mathcal{X}$, respectively, and B_E and B_F be their compatibility operators of type $E^1 \to E^2$ and $F^1 \to F^2$. Then the complexes*

$$\begin{array}{ccccc} \mathcal{E}(\mathcal{X}, J(E^0)) & \stackrel{A_E}{\to} & \mathcal{E}(\mathcal{X}, J(E^1)) & \stackrel{B_E}{\to} & \mathcal{E}(\mathcal{X}, J(E^2)) \\ \mathcal{E}(\mathcal{X}, J(F^0)) & \stackrel{A_F}{\to} & \mathcal{E}(\mathcal{X}, J(F^1)) & \stackrel{B_F}{\to} & \mathcal{E}(\mathcal{X}, J(F^2)) \end{array}$$

are homotopy equivalent.

Proof. By Lemma 6.3, the complexes $\{A_E, B_E\}$ and $\{A_F, B_F\}$ are homotopy equivalent. This means, there exist differential operators M_i of type $F^i \to E^i$ and M_i^{-1} of type $E^i \to F^i$, for $i = 0, 1, 2$, and differential operators h_i^E of type $E^i \to E^{i-1}$ and h_i^F of type $F^i \to F^{i-1}$, for $i = 1, 2$, with the property that the following conditions are fulfilled:

$$\begin{array}{lrcllrcl} 1) & M_{i+1}A_F^i - A_E^i M_i & = & 0, & 2) & M_i^{-1}M_i & = & I - h_{i+1}^F A_F^i - A_F^{i-1} h_i^F, \\ & M_{i+1}^{-1}A_E^i - A_F^i M_i^{-1} & = & 0; & & M_i M_i^{-1} & = & I - h_{i+1}^E A_E^i - A_E^{i-1} h_i^E \end{array}$$

for $i = 0, 1$, where $A_E^0 = A_E$, $A_E^1 = B_E$ and $A_F^0 = A_F$, $A_F^1 = B_F$. We now apply Lemma 4.3 to obtain

$$\begin{array}{lrcl} 1) & h(j^\infty M_{i+1})h(j^\infty A_F^i) - h(j^\infty A_E^i)h(j^\infty M_i) & = & 0, \\ & h(j^\infty M_{i+1}^{-1})h(j^\infty A_E^i) - h(j^\infty A_F^i)h(j^\infty M_i^{-1}) & = & 0; \end{array}$$

$$\begin{array}{lrcl} 2) & h(j^\infty M_i^{-1})h(j^\infty M_i) & = & I - h(j^\infty h_{i+1}^F)h(j^\infty A_F^i) - h(j^\infty A_F^{i-1})h(j^\infty h_i^F), \\ & h(j^\infty M_i)h(j^\infty M_i^{-1}) & = & I - h(j^\infty h_{i+1}^E)h(j^\infty A_E^i) - h(j^\infty A_E^{i-1})h(j^\infty h_i^E) \end{array}$$

for $i = 0, 1$. This shows immediately that the complexes of bundle homomorphisms $\{h(j^\infty A_E), h(j^\infty B_E)\}$ and $\{h(j^\infty A_F), h(j^\infty B_F)\}$ are homotopy equivalent, which is the desired conclusion. □

In particular, both the complexes have the same cohomology, see for instance Corollary 1.1.14 in [Tar95].

We next extend Theorem 5.6 to the case $s = \infty$. The proof given above does not go in the case $s = \infty$, for in no way it is obvious that the columns in (5.5) are exact.

Theorem 9.2. *Let A be a sufficiently regular differential operator on $\mathcal{X}$ and B a compatibility operator for A. Then, if U is sufficiently small, for each formal power series $f \in \mathcal{E}(U, J(F))$ satisfying $Bf = 0$ in U there exists a formal power series $u \in \mathcal{E}(U, J(E))$ with $Au = f$.*

Proof. In view of Theorem 6.2 and Lemmas 6.3 and 9.1 we may assume without loss of generality that:

1) U is a coordinate neighbourhood in $\mathcal{X}$ over which the bundles E and F are trivial.
2) A is a normalised operator of the form (6.2).
3) Commutativity relations hold in (6.2) for coordinates $x = (x^1, \dots, x^n)$ in U.
4) The compatibility operator B for A is given by (6.4).

Write

$$A = \sum_{j=1}^{n} A_j(x)\partial_j + A_0(x),$$

for $x \in U$. We now invoke induction in $m \in \{1, \dots, n\}$, the number of non-zero coefficients $A_j(x)$. We can assume, by renumbering the coefficients if necessary, that the non-zero coefficients are $A_{n-m+1}(x), \dots, A_n(x)$.

If $m = 1$, then we argue as follows. By Definition 6.1, the system $Au = 0$ does not contain purely algebraic relations between components $(u_1, \dots, u_k)$ of u. More precisely, A has the form

$$Au = \Big(\partial_n + T^{(1)}(x)\Big)\, u^{(1)} + T^{(2)}(x, \partial_n) u^{(2)}$$

for $u \in \mathcal{E}(U, E)$, cf. (6.2).

Since both $M^{(1)}$ and $M^{(2)}$ vanish for $m = 1$, we see that $B = 0$ in this case. Hence, the desired result follows immediately from Theorem 8.1. Indeed, choose $u^{(2)} \in \mathcal{E}(U, J(\mathbb{C}^{k_2}))$ and the data $u^{(1,0)}_{\alpha'} \in \mathcal{E}(U, \mathbb{C}^{k_1})$, for $\alpha' \in \mathbb{N}_0^{n-1}$, in an arbitrary way. Then we apply Theorem 8.1 to the operator $D = \partial_n + T^{(1)}(x)$, when considering the Cauchy problem

$$\begin{array}{rcl} h(j^\infty D) u^{(1)}\,(x,z) & = & f(x,z) - h(j^\infty T^{(2)}) u^{(2)}\,(x,z), \\ u^{(1)}_{(\alpha',0)}(x) & = & u^{(1,0)}_{\alpha'}(x), \quad \alpha' \in \mathbb{N}_0^{n-1}, \end{array}$$

for $x \in U$, cf. (8.1). As a result we get a unique solution $u^{(1)} \in \mathcal{E}(U, J(\mathbb{C}^{k_1}))$ of this problem. By the very construction, $u = u^{(1)} \oplus u^{(2)}$ belongs to $\mathcal{E}(U, J(E))$ and satisfies $Au = f$.

For $m > 1$, the operator $M = (M^{(1)}, M^{(2)})$ of (6.2) contains the derivatives in $x^{n-m+1}, \dots, x^{n-1}$ only. The inductive hypothesis allows us to assume that the complex $\{M, N\}$ is exact on the level of formal power series over U. More precisely, let $f \in \mathcal{E}(U, J(F))$ satisfy $h(j^\infty B) f = 0$. When writing $f = f^{(1)} \oplus f^{(2)}$ with components $f^{(i)} \in \mathcal{E}(U, J(\mathbb{C}^{\ell_i}))$, $i = 1, 2$, we obtain $h(j^\infty N) f^{(1)} = 0$ in U. Hence, there exists a formal power series $v \in \mathcal{E}(U, J(E))$ with the property that $h(j^\infty M) v = f^{(1)}$.

We now write $v = v^{(1)} \oplus v^{(2)}$ in accordance with the bundle decomposition $E_p \cong \mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$ over U. Denote by D the differential operator in the lower left corner of A, i.e.,

$$D = \partial_n + T^{(1)}(x, \partial_{n-m+1}, \dots, \partial_{n-1}).$$

By Theorem 8.1, there is a unique formal power series $u^{(1)} \in \mathcal{E}(U, J(\mathbb{C}^{k_1}))$ solving the Cauchy problem

$$\begin{array}{rcl} h(j^\infty D) u^{(1)}\,(x,z) & = & f^{(2)}(x,z) - h(j^\infty T^{(2)}) v^{(2)}\,(x,z), \\ u^{(1)}_{(\alpha',0)}(x) & = & v^{(1)}_{\alpha'}(x), \quad \alpha' \in \mathbb{N}_0^{n-1}, \end{array}$$

for $x \in U$, cf. (8.1). Set $u^{(2)} = v^{(2)}$. By construction, the sum $u = u^{(1)} \oplus u^{(2)}$ is in $\mathcal{E}(U, J(E))$ and satisfies $(\partial_n + T^{(1)}) u^{(1)} + T^{(2)} u^{(2)} = f^{(2)}$.

Our next claim is that $Mu = f^{(1)}$, the action of M being identified with that of $h(j^\infty M)$. To prove this, we observe that commutativity relations (6.3) just amount to

$$(\partial_n + S^{(1)}) Mu = M^{(1)}((\partial_n + T^{(1)}) u^{(1)} + T^{(2)} u^{(2)})$$

for all $u \in \mathcal{E}(U, J(E))$. Since $Bf = 0$ in U, we get $(\partial_n + S^{(1)})f^{(1)} = M^{(1)} f^{(2)}$, and so

$$\begin{aligned}(\partial_n + S^{(1)})(Mu - f^{(1)}) &= (\partial_n + S^{(1)})Mu - (\partial_n + S^{(1)})f^{(1)} \\ &= M^{(1)}((\partial_n + T^{(1)})u^{(1)} + T^{(2)}u^{(2)}) - M^{(1)} f^{(2)} \\ &= 0.\end{aligned}$$

By construction, the coefficients $(u^{(1)} - v^{(1)})_{(\alpha',0)}$ vanish in U for all multi-indices $\alpha' \in \mathbb{N}_0^{n-1}$. Moreover, from $u^{(2)} = v^{(2)}$ it follows that

$$\begin{aligned}(Mu - f^{(1)})_{(\alpha',0)} &= (Mu - Mv)_{(\alpha',0)} \\ &= (M^{(1)}(u^{(1)} - v^{(1)}))_{(\alpha',0)}. \end{aligned} \tag{9.1}$$

Since the operator $M^{(1)}$ contains the derivatives in $x^{n-m+1}, \ldots, x^{n-1}$ only, we deduce from (9.1) that

$$(Mu - f^{(1)})_{(\alpha',0)} = 0$$

for all $\alpha' \in \mathbb{N}_0^{n-1}$.

Finally, the solution of the Cauchy problem for the operator $\partial_n + S^{(1)}$ in the class of formal power series is unique, which is due to Theorem 8.1. Therefore, $Mu - f^{(1)} = 0$ in U, as desired. □

If the coefficients of the operator A and the right-hand side f are real analytic in U then among the formal solutions of $Au = f$ in U constructed in Theorem 9.2 there are also real analytic ones, see for instance Theorem 1.3.40 of [Tar95]. To construct such a solution, one has to choose "proper" real analytic data for the Cauchy problem and use the Cauchy-Kovalevskaya theorem instead of its formal version given by Theorem 8.1.

Lemma 9.3. *Let $s \in \mathbb{N}_0 \cup \{\infty\}$ satisfy $s \geq a$. Assume that $u \in \mathcal{E}(\mathcal{X}, J^s(E))$, $f \in \mathcal{E}(\mathcal{X}, J^{s-a}(F))$ and $h(j^{s-a}A)u = f$. Then f is $(s-a)$-jet of some section in $\mathcal{E}(\mathcal{X}, F)$ if and only if*

$$(h(j^{s-1-a}A) \otimes I)\mathfrak{d}^s u = 0.$$

Proof. Indeed, under the hypothesis of the lemma, Lemma 4.4 implies that

$$\begin{aligned}(h(j^{s-1-a}A) \otimes I)\mathfrak{d}^s u &= \mathfrak{d}^{s-a}\, h(j^{s-a}A)u \\ &= \mathfrak{d}^{s-a} f.\end{aligned}$$

Since f stems from some section in $\mathcal{E}(\mathcal{X}, F)$ if and only if $\mathfrak{d}^{s-a} f = 0$, the lemma follows. □

When combined with Lemma 9.3, Theorem 9.2 implies that the cohomology of (5.3) depends on the structure of the space of solutions to the homogeneous equation $h(j^\infty A)u = 0$. Indeed, if f is a formal power series of some section in $\mathcal{E}(U, F)$ satisfying $Bf = 0$, then the solution $u \in \mathcal{E}(U, J(E))$ given by Theorem 9.2 is not arbitrary. Namely, the image $\mathfrak{d}u$ of u by the connection proves to belong to $\Omega^1(U, \mathcal{R}^\infty)$.

Recall that by $\mathcal{R}^\infty$ is meant the null-space of the vector bundle homomorphism $h(j^\infty A) : J(E) \to J(F)$. This fibre space over $\mathcal{X}$ need not behave well unless A is a sufficiently regular differential operator. In the latter case $\mathcal{R}^\infty$ is a vector subbundle of generically infinite rank in $J(E)$. We are now in a position to extend Theorem 5.6 to the case $s = \infty$.

Theorem 9.4. *Let $A \in \mathrm{Diff}^a(\mathcal{X}; E, F)$ be a sufficiently regular differential operator and $\{A^i\}_{i=0,1,\dots}$ a compatibility complex for A. Then the cohomologies of the complexes*

$$\begin{array}{l} 0 \to \mathcal{S}_{\ker A}(U) \overset{j^\infty}{\to} \mathcal{E}(U, \mathcal{R}^\infty) \overset{\mathfrak{d}}{\to} \Omega^1(U, \mathcal{R}^\infty) \overset{\mathfrak{d}}{\to} \dots \overset{\mathfrak{d}}{\to} \Omega^n(U, \mathcal{R}^\infty) \to 0, \\ 0 \to \mathcal{S}_{\ker A}(U) \overset{\hookrightarrow}{\to} \mathcal{E}(U, E^0) \overset{A^0}{\to} \mathcal{E}(U, E^1) \overset{A^1}{\to} \dots \overset{A^{n-1}}{\to} \mathcal{E}(U, E^n) \to \dots \end{array}$$

are the same, provided that U is small enough.

Proof. The relationship between the complexes in question is expressed by the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{S}_{\ker A}(X) & \overset{j^\infty}{\to} & \mathcal{E}(U, \mathcal{R}^\infty) & \overset{\mathfrak{d}}{\to} & \Omega^1(U, \mathcal{R}^\infty) & \overset{\mathfrak{d}}{\to} & \dots \\ & & \Big\downarrow{\scriptstyle \hookrightarrow} & & \Big\downarrow{\scriptstyle \hookrightarrow} & & \Big\downarrow{\scriptstyle \hookrightarrow} & & \\ 0 & \to & \mathcal{E}(U, E^0) & \overset{j^\infty}{\to} & \mathcal{E}(U, J(E^0)) & \overset{\mathfrak{d}}{\to} & \Omega^1(U, J(E^0)) & \overset{\mathfrak{d}}{\to} & \dots \\ & & \Big\downarrow{\scriptstyle A^0} & & \Big\downarrow{\scriptstyle h(j^\infty A^0)} & & \Big\downarrow{\scriptstyle h(j^\infty A^0)\otimes I} & & \\ 0 & \to & \mathcal{E}(U, E^1) & \overset{j^\infty}{\to} & \mathcal{E}(U, J(E^1)) & \overset{\mathfrak{d}}{\to} & \Omega^1(U, J(E^1)) & \overset{\mathfrak{d}}{\to} & \dots \\ & & \Big\downarrow{\scriptstyle A^1} & & \Big\downarrow{\scriptstyle h(j^\infty A^1)} & & \Big\downarrow{\scriptstyle h(j^\infty A^1)\otimes I} & & \\ & & \dots & & \dots & & \dots & & \end{array} \tag{9.2}$$

which commutes. From the exactness of the first Spencer sequence for the trivial operator and Theorem 9.2 we deduce that the rows and columns in the diagram are exact except possibly for the first row and first column. Thus by diagram chasing the cohomology of the first row is the same as the cohomology of the first column. □

One may ask whether Theorem 9.4 is still true if $U = \mathcal{X}$ but we will not develop this point here.

Similarly to (5.4) we get the complex of sheaves

$$0 \to \mathcal{S}_{\ker A} \overset{j^\infty}{\to} \mathcal{S}_{\mathcal{R}^\infty} \overset{\mathfrak{d}}{\to} \mathcal{S}_{\mathcal{R}^\infty \otimes \Lambda^1} \overset{\mathfrak{d}}{\to} \dots \overset{\mathfrak{d}}{\to} \mathcal{S}_{\mathcal{R}^\infty \otimes \Lambda^n} \to 0 \tag{9.3}$$

over $\mathcal{X}$, which will be referred to as the limit first sequence of Spencer for the operator A.

Corollary 9.5. *Let $A^i \in \mathrm{Diff}^{a_i}(\mathcal{X}; E^i, E^{i+1})$, $i = 0, 1, \dots$, be a compatibility complex for a sufficiently regular differential operator $A = A^0$. Then the cohomology of the complex*

$$0 \to \mathcal{S}_{\ker A} \overset{\hookrightarrow}{\to} \mathcal{S}_{E^0} \overset{A}{\to} \mathcal{S}_{E^1} \overset{A}{\to} \dots \overset{A}{\to} \mathcal{S}_{E^n} \to \dots$$

coincides with the cohomology of the limit first sequence of Spencer for A..

Proof. This is an immediate consequence of Theorem 9.4. □

We have thus reduced the problem on solvability of an overdetermined system of differential equations with smooth coefficients to the following one. Under what conditions does a connection $\mathfrak{d}$ of zero curvature on a vector bundle $\mathcal{R}$ of infinite rank give rise to Fredholm complexes of differential forms with coefficients in the bundle $\mathcal{R}$?

Of course, we can consider the limit first sequence of Spencer also for those differential operators which possess no regularity property. However, in this case this sequence need not bear any information about the cohomology of the initial complex.

Example 9.6. Let $Au := au$ be the operator of Example 2.3 and $U = \mathbb{R}$. Then $u \in \mathcal{E}(U, \mathcal{R}^\infty)$ if and only if

$$\sum_{\gamma=0}^{\alpha} \frac{a^{(\alpha-\gamma)}(x)}{(\alpha-\gamma)!} u_\gamma(x) = 0$$

in U for all $\alpha \in \mathbb{N}_0$. Obviously, this holds if and only if $a(x)u_\gamma(x) = 0$ for all $x \in U$ and $\gamma \in \mathbb{N}_0$, i.e., $u_\gamma \in \mathcal{S}_{\ker A}(U)$. Thus, we can identify $\mathcal{E}(U, \mathcal{R}^\infty)$ with the product of countably many copies of $\mathcal{S}_{\ker A}(U)$. Now we easily see that the limit first sequence of Spencer for the operator A

$$0 \to \mathcal{S}_{\ker A}(U) \stackrel{j^\infty}{\to} \mathcal{E}(U, \mathcal{R}^\infty) \stackrel{\mathfrak{d}}{\to} \Omega^1(U, \mathcal{R}^\infty) \to 0$$

is exact over U. Indeed, the exactness at steps 0 and 1 has already been discussed. As to exactness at step 2, we note that each $f \in \Omega^1(U, \mathcal{R}^\infty)$ has the form

$$f(x,z) = \Big(\sum_{\alpha \in \mathbb{N}_0} f_\alpha(x) z^\alpha \Big) dx$$

in U. Take

$$\begin{array}{rcl} u_0 & = & 0, \\ u_\alpha & = & u'_{\alpha-1} - f_{\alpha-1} \end{array}$$

for $\alpha \geq 1$. Since $u'_{\alpha-1}$ belongs to $\mathcal{S}_{\ker A}(U)$ if so does $u_{\alpha-1}$, we conclude that $u \in \mathcal{E}(U, \mathcal{R}^\infty)$ and $\mathfrak{d}u = f$, as desired. On the other hand, the operator A itself does not admit a compatibility complex on the level of sheaves of germs of smooth functions over U at all..

10. Holonomic systems

In this section we treat overdetermined systems maximally closed to systems of ordinary differential equations.

Definition 10.1. A differential operator A of type $E \to F$ on $\mathcal{X}$ is said to be holonomic if:

1) A is sufficiently regular.
2) There is $Q \in \mathbb{N}_0$ such that the symbolic bundles σ^s of the $(s-a)$ th prolongation $j^{s-a} \circ A$ are zero for all $s \geq Q$.

Roughly speaking, a holonomic system is a highly overdetermined system, such that the solutions locally form a vector space of finite dimension, instead of the expected dependence on some arbitrary function. Such systems have been applied, for example, to the Riemann-Hilbert problem in higher dimensions, and to quantum field theory, cf. [Kas75, Kas78].

Theorem 10.2. *Let A be holonomic and U be a simply connected small domain. Then for each $f \in \mathcal{E}(U, F)$ satisfying $Bf = 0$ in U there exists $u \in \mathcal{E}(U, E)$ with $Au = f$.*

Another way of stating this theorem is to say: a C^∞ Poincaré lemma holds for holonomic systems.

Proof. Since A is holonomic, the bundle homomorphism $\pi^{s-1,s} : \mathcal{R}^s \to \mathcal{R}^{s-1}$ is actually an isomorphism for sufficiently large s. This means, in particular, that the bundle $\mathcal{R}^\infty$ is isomorphic to $\mathcal{R}^Q$, and so it has a finite rank $r \geq 0$. Hence, for each $p \in \mathcal{X}$ we can find a neighbourhood U of p and a finite basis $\{b_1(x), \ldots, b_r(x)\}$ in $\mathcal{E}(U, \mathcal{R}^\infty)$..

More precisely, we mean the following:

1) The system $\{b_1(x), \ldots, b_r(x)\}$ is linearly independent over the ring $\mathcal{E}(U)$.

2) For every $u \in \mathcal{E}(U, \mathcal{R}^\infty)$ there are (unique) coefficients $c^k \in \mathcal{E}(U)$ with the property that

$$u(x, z) = \sum_{k=1}^{r} c^k(x) b_k(x, z)$$

whenever $x \in U$.

From 2) it follows that for any $u \in \Omega^q(U, \mathcal{R}^\infty)$ there are (unique) differential forms $c^k \in \Omega^q(U)$, such that

$$u(x, z) = \sum_{k=1}^{r} c^k(x) b_k(x, z)$$

whenever $x \in U$. Then

$$\mathfrak{d}^q u\,(x, z) = \sum_{k=1}^{r} dc^k(x)\, b_k(x, z) + (-1)^q c^k(x) \wedge \mathfrak{d}^0 b_k(x, z)$$

for all $x \in U$. As $\mathfrak{d}^0 b_k \in \Omega^1(U, \mathcal{R}^\infty)$, we conclude that there are differential forms $t_k^j \in \Omega^1(U)$ with

$$\mathfrak{d}^0 b_k(x, z) = \sum_{j=1}^{r} t_k^j(x) b_j(x, z).$$

Note that $\mathfrak{d}^1 \mathfrak{d}^0 = 0$ implies

$$dt_k^j(x) - \sum_{\ell=1}^{r} t_\ell^j(x) \wedge t_k^\ell(x) = 0 \tag{10.1}$$

for all $x \in U$ and $1 \leq j, k \leq r$.

Thus, under the local "basis" $\{b_1(x), \ldots, b_r(x)\}$ the operator $\mathfrak{d}^q$ may be represented as $\tilde{\mathfrak{d}}^q$ mapping $\Omega^q(U)^r$ to $\Omega^{q+1}(U)^r$ by

$$(\tilde{\mathfrak{d}}^q c)^j = dc^j + \sum_{k=1}^{r} t_k^j \wedge c^k,$$

c being an r-column with components $c^1, \ldots, c^r$. By the very construction, the new complex $\{\Omega^q(U)^r, \tilde{\mathfrak{d}}^q\}_{q=0,1,\ldots,n}$ is equivalent to the first Spencer sequence and so it gives us a compatibility complex for $\tilde{\mathfrak{d}}^0$.

For $q = 0$, we get

$$(\tilde{\mathfrak{d}}^0 c)^j = dc^j + \sum_{\ell=1}^{n} \Big(\sum_{k=1}^{r} t^j_{k,\ell}\, c^k \Big) dx^\ell, \tag{10.2}$$

which gives rise to a Frobenius type system, see for instance [Har64]. □

11. A homotopy operator

Since $(\mathfrak{d}^s u)(x,z) = du(x, z-x)$ for $s = \infty$, one can try to invoke the classical Poincaré-Cartan homotopy operator, to treat the inhomogeneous equation $\mathfrak{d}u = f$ for any $f \in \Omega^q(U, \mathcal{R}^\infty)$ satisfying $\mathfrak{d}f = 0$. Let U is a star-shaped domain in $\mathbb{R}^n$. There is no loss of generality in assuming that U is star-shaped with respect to the origin.

Given any

$$f(x, z-x) = {\sum_{\#I=q}}' \Big(\sum_{\alpha \in \mathbb{Z}_0^n} f_{I,\alpha}(x)(z-x)^\alpha \Big) dx^I,$$

the homotopy operator of Poincaré-Cartan is

$$\begin{aligned}(h^q f)(x, z-x) &= \int_0^1 dt \rfloor f(tx, z-tx)\, dt \\ &= {\sum_{\#I=q}}' \Big(\int_0^1 t^{q-1} \sum_{\alpha \in \mathbb{Z}_0^n} f_{I,\alpha}(tx)(z-tx)^\alpha\, dt \Big) \iota(X) dx^I\end{aligned}$$

for $(x,z) \in U \times \mathbb{C}^n$, where

$$\iota(X) dx^I = \sum_{k=1}^{q} (-1)^{k-1} x^{i_k} dx^I[i_k]$$

stands for the interior product of the differential form dx^I by the vector field $X = x^1 \partial_1 + \ldots + x^n \partial_n$, and $dx^I[i_k]$ is the exterior product of the differentials $dx^{i_1}, \ldots, dx^{i_q}$ with the exception of dx^{i_k}.

Using the binomial formula

$$(z-tx)^\alpha = \sum_{\beta \le \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} (z-x)^{\alpha-\beta} ((1-t)x)^\beta,$$

we introduce $(h^q f)(x,z) = 0$, if $q = 0$, and

$$\begin{aligned}(h^q f)(x,z) &= {\sum_{\#I=q}}' \Big(\sum_{\substack{\alpha \in \mathbb{Z}_0^n \\ \beta \le \alpha}} \frac{\alpha!}{(\alpha-\beta)!\beta!} \int_0^1 t^{q-1} f_{I,\alpha}(tx)((1-t)x)^\beta\, dt\, z^{\alpha-\beta} \Big) \iota(X) dx^I \\ &= {\sum_{\#I=q}}' \Big(\sum_{\gamma \in \mathbb{Z}_0^n} c_{I,\gamma}(f) z^\gamma \Big) \iota(X) dx^I,\end{aligned} \tag{11.1}$$

if $q \ge 1$, where

$$c_{I,\gamma}(f)(x) = \sum_{\beta \in \mathbb{Z}_0^n} \Big(\frac{(\gamma+\beta)!}{\gamma!\beta!} \int_0^1 t^{q-1} (1-t)^{|\beta|} f_{I,\gamma+\beta}(tx) dt \Big) x^\beta.$$

The second line on the right-hand side of (11.1) just amounts to the formal (sic!) homotopy operator for the complex (9.3) constructed by Buttin [But67]. The homotopy operator is formal, for the coefficients $c_{I,\gamma}(f)$ are actually formal series in the powers of x which need not converge. She proved that hf makes sense if the components of the jet f are real analytic and satisfy the Cauchy inequality in U, i.e., if f corresponds to a real analytic solution to the equation $Af_0 = 0$ in U. Moreover, the operator h obeys the structure of $\Omega^q(U, \mathcal{R}^\infty)$, at least formally. She also illustrated how the formal homotopy operator h may be used to obtain an easy proof of the analytic Poincaré lemma for formally exact complexes of differential operators with real analytic coefficients and a C^∞ Poincaré lemma for elliptic complexes of such operators. We wish to extend this construction to the C^∞ case. To this end we observe that the first line in (11.1) makes sense for all C^∞ functions $f_{I,\alpha}$ in U as certain rearrangement of a formal power series in z whose coefficients are themselves series of functions which diverge in general. It cannot therefore be specified within infinite jets, and so it is no longer an element of $\Omega^{q-1}(U, \mathcal{R}^\infty)$. We thus arrive at objects of more general structure which allow one to solve the equation $\mathfrak{d}u = f$.

Theorem 11.1. *Suppose U is a domain in $\mathbb{R}^n$ star-shaped with respect to the origin. Then, for all $f \in \Omega^q(U, \mathcal{R}^\infty)$ with $q \geq 1$, we have*

$$h^{q+1}\mathfrak{d}^q f + \mathfrak{d}^{q-1} h^q f = f \tag{11.2}$$

in U. Moreover, $u = h^q f$ belongs to $\Omega^{q-1}(U, \mathcal{R}^\infty)$ provided that the relevant series converge.

Proof. This readily follows from the homotopy formula of Poincaré-Cartan once we allow formal computations. □

We finish this section by analysing the homotopy formula (11.2) in the case of operator $Au = du - au$ acting on scalar-valued functions in a star-shaped domain $U \subset \mathbb{R}^n$, where $a \in \Omega^1(U)$. The system $du - au = f$ is obviously of Frobenius type with $r = 1$.

It follows from formula (4.7) that $u \in J^\infty_x(U \times \mathbb{C})$ belongs to $\mathcal{R}^\infty(x)$ if and only if

$$u_{\alpha+e_j} = \frac{1}{\alpha_j + 1} \sum_{\gamma \leq \alpha} \frac{\partial^{\alpha-\gamma} a_j(x)}{(\alpha - \gamma)!} u_\gamma \tag{11.3}$$

for all $\alpha \in \mathbb{Z}^n_0$ and $1 \leq j \leq n$.

It is easy to see that for the system (11.3) to possess a nontrivial solution at the point $x \in U$ it is necessary and sufficient that

$$\frac{\partial}{\partial x^k} a_j(x) = \frac{\partial}{\partial x^j} a_k(x) \tag{11.4}$$

for all $1 \leq j, k \leq n$. If (11.4) is not fulfilled, the rank of $\mathcal{R}^\infty(x)$ is therefore equal to zero.

If (11.4) holds then we prove by induction that a solution to (11.3) is of the form $u = u_0 b(x)$, where $b(x) = (1, a_1(x), \ldots, a_n(x), \ldots)$ is independent on u_0. Hence, the rank of $\mathcal{R}^\infty(x)$ is equal to one in this case.

Thus, the operator $A = d - a$ is sufficiently regular over U merely in two cases: 1) (11.4) is satisfied for all $x \in U$; 2) (11.4) does not fulfil at any point $x \in U$. Obviously, the condition 1) just amounts to saying that $da(x)$ is different from zero for all $x \in U$. The condition 2) is equivalent to the fact that $da(x) = 0$ for all $x \in U$. Note that the latter equality precisely coincides with (10.1) for $r = 1$ with $t^1_1 = -a$.

In case $da = 0$ in U the Poincaré-Cartan operator is well defined not only formally. In this case we can easily specify the basis in $\mathcal{R}^\infty(x)$. More precisely, we claim that $u \in \mathcal{R}^\infty(x)$ if and only if

$$u_\alpha = u_0\, e^{-\int^x a} \frac{1}{\alpha!} \partial^\alpha e^{\int^x a} \tag{11.5}$$

for all $\alpha \in \mathbb{Z}_0^n$, where $\int^x a$ is a primitive function for a in a neighbourhood of x. To prove this, we argue by induction.

For $\alpha = 0$ the equality is obvious. For $\alpha \in \mathbb{Z}_0^n$ of norm 1 formula (11.3) readily implies $u_{e_j} = a_j(x)$ for $j = 1, \dots, n$, as desired. If now (11.5) is true for all $\alpha \in \mathbb{Z}_0^n$ of norm $\le s$, then for $|\alpha| = s$ we get

$$\begin{aligned} u_{\alpha+e_j} &= u_0\, e^{-\int^x a} \frac{1}{\alpha_j + 1} \sum_{\gamma \le \alpha} \frac{1}{(\alpha-\gamma)!\gamma!}\, \partial^{\alpha-\gamma} a_j(x)\, \partial^\gamma e^{\int^x a} \\ &= u_0\, e^{-\int^x a} \frac{1}{(\alpha+e_j)!} \partial^\alpha \left(a_j(x) e^{\int^x a} \right) \\ &= u_0\, e^{-\int^x a} \frac{1}{(\alpha+e_j)!} \partial^{\alpha+e_j} e^{\int^x a} \end{aligned}$$

by (11.3), which was to be proved.

From what has been proved it follows that the non-zero jet $j^\infty(e^{\int^x a})(x,z)$, $x \in U$, forms a basis in $\mathcal{E}(U, \mathcal{R}^\infty)$ over the ring $\mathcal{E}(U)$.

Therefore, for every $f \in \Omega^q(U, \mathcal{R}^\infty)$ there is a differential form $c \in \Omega^q(U)$ with the property that $f(x,z) = c(x) j^\infty(e^{\int^x a})(x,z)$ for $x \in U$. Substituting this into (11.1) yields

$$\begin{aligned} &(h^q f)(x,z) \\ = &\sum_{\#I=q}{}' \Big(\sum_{\gamma \in \mathbb{Z}_0^n} \int_0^1 t^{q-1} c_I(tx) \frac{1}{\gamma!} \sum_{\beta \in \mathbb{Z}_0^n} \frac{1}{\beta!} \partial^\beta \left(\partial^\gamma e^{\int^x a} \right)(tx)(x-tx)^\beta dt z^\gamma \Big) \iota(X) dx^I \\ = &\sum_{\#I=q}{}' \Big(\sum_{\gamma \in \mathbb{Z}_0^n} \int_0^1 t^{q-1} c_I(tx)\, j^\infty \Big(\frac{\partial^\gamma e^{\int^x a}}{\gamma!} \Big)(tx, x-tx) dt\, z^\gamma \Big) \iota(X) dx^I \end{aligned}$$

for $q \ge 1$.

If the differential form a is real analytic on U then its primitive $\int^x a$ is real analytic, too, and so

$$\begin{aligned} (h^q f)(x,z) &= \sum_{\#I=q}{}' \Big(\int_0^1 t^{q-1} c_I(tx)\, dt \Big) \iota(X) dx^I\, j^\infty \left(e^{\int^x a} \right)(x,z) \\ &= (h^q c)(x)\, j^\infty \left(e^{\int^x a} \right)(x,z) \end{aligned}$$

for all $x \in U$. This fact hints us to introduce a "regularisation" of the Buttin homotopy operator by $\tilde{h}^0 f = 0$ and $(\tilde{h}^q f) = (h^q c)\, j^\infty(e^{\int^x a})$, if $q > 0$. By the very construction, $\tilde{h}^q$ maps $\Omega^q(U, \mathcal{R}^\infty)$ to $\Omega^{q-1}(U, \mathcal{R}^\infty)$. As $\mathfrak{d}(c\, j^\infty u) = dc\, j^\infty u$, we easily deduce that

$$\tilde{h}^{q+1} \mathfrak{d}^q f + \mathfrak{d}^{q-1} \tilde{h}^q f = f$$

for $q \ge 1$.

Since $(da)u = a \wedge (Au) - d(Au)$, we arrive at the following topic of E. Cartan theory.

If da is different from zero at each point of U, then the equation $Au = f$ is solvable in $\mathcal{E}(U)$ if and only if $f \in \Omega^1(U)$ satisfies

$$(\partial_j a_i - \partial_i a_j)((\partial_\ell - a_\ell) f_k - (\partial_k - a_k) f_\ell) = (\partial_\ell a_k - \partial_k a_\ell)((\partial_j - a_j) f_i - (\partial_i - a_i) f_j)$$

for all integers $1 \le i, j, k, \ell \le n$. Moreover, the solution is unique and it is given by the formula

$$u = \frac{(\partial_j - a_j) f_i - (\partial_i - a_i) f_j}{\partial_j a_i - \partial_i a_j}$$

provided $i \ne j$. In particular, $u \equiv 0$ if $f \equiv 0$.

If the differential form a is closed in U, then the equation $Au = f$ is solvable in $\mathcal{E}(U)$ if and only if $f \in \Omega^1(U)$ fulfills $df - a \wedge f = 0$ in U. Moreover, the function family

$$u(x) = e^{\int^x a} \Big(\sum_{j=1}^n x_j \int_0^1 e^{-\int^{tx} a} f_j(tx) dt + \mathrm{const} \Big)$$

is a general solution to the inhomogeneous equation $Au = f$. In particular, we get $u = c\, e^{\int^x a}$ if $f = 0$.

We thus conclude that in both cases the space $\mathcal{E}(U, \mathcal{R}^\infty)$ is generated by the set $j^\infty(\mathcal{S}_{\ker A}(U))$ over the ring $\mathcal{E}(U)$.

Remark 11.2. For $a = x_2^3 dx_1 - x_1^3 dx_2$ in $\mathbb{R}^2$ we have $da = -3|x|^2 dx$, and so A is not sufficiently regular in any neighbourhood U of the origin. The solution to $Au = f$ in $U \setminus \{0\}$ is given by

$$u = \frac{(\partial_2 + x_1^3) f_1 - (\partial_1 - x_2^3) f_2}{3(x_1^2 + x_2^2)},$$

hence the conditions for the existence of a smooth solution in U are obvious. However, they are not differential.

12. Frobenius type systems

We begin by discussing overdetermined differential operators with null-spaces of finite dimension whose compatibility complexes are essentially closed to the de Rham complex.

To this end, we recall the so-called uniqueness condition for the Cauchy problem in the small on $\mathcal{X}$, denoted by $(U)_s$: Given any connected open set $U \subset \mathcal{X}$, if $u \in \mathcal{S}_{\ker A}(U)$ vanishes on a nonempty open subset of U then $u \equiv 0$ on all of U. This property implies in particular the existence of a left fundamental solution for A on $\mathcal{X}$.

Lemma 12.1. *Suppose U is a connected open set in $\mathcal{X}$. If a differential operator A satisfies the uniqueness condition $(U)_s$, then, for any linearly independent system $\{u_i\}_{i \in I}$ in $\mathcal{S}_{\ker A}(U)$, the system $\{j^\infty(u_i)\}_{i \in I}$ is linearly independent over the ring $\mathcal{E}(U)$.*

Proof. If a solution $u \in \mathcal{S}_{\ker A}(U)$ satisfies $c(x) j^\infty(u)(x, z) = 0$ for all $x \in U$ with a smooth function $c \ne 0$ in U, then u vanishes on a nonempty open subset of U. By the unique continuation property we readily conclude that $u = 0$ on all of U. Hence, the jet $j^\infty(u)$ of any non-zero solution $u \in \mathcal{S}_{\ker A}(U)$ is always linearly independent over $\mathcal{E}(U)$.

Let now $\{u_i\}_{i \in I}$ be a linearly independent system in the space $\mathcal{S}_{\ker A}(U)$. It follows that $u_i \not\equiv 0$ in U for all $i \in I$. By the above, each subsystem of $\{j^\infty(u_i)\}_{i \in I}$

consisting of one element is linearly independent over $\mathcal{E}(U)$. We next argue by induction.

Assume that all subsystems of $\{j^\infty(u_i)\}_{i\in I}$ containing at most N elements are linearly independent over $\mathcal{E}(U)$. Pick a system $\{j^\infty(u_{i_1}),\dots,j^\infty(u_{i_{N+1}})\}$ with $i_1,\dots,i_{N+1}\in I$. Let

$$\sum_{\nu=1}^{N+1} c_\nu(x)\, j^\infty(u_{i_\nu})(x,z)=0$$

for all $x\in U$, where c_ν are smooth functions in U which are not all identically zero in U. Without loss of generality we can assume that $c_{N+1}(x)\equiv -1$ on a nonempty open set $U_1\subset U$, i.e.,

$$\partial^\alpha u_{i_{N+1}}(x)=\sum_{\nu=1}^{N} c_\nu(x)\,\partial^\alpha u_{i_\nu}(x) \tag{12.1}$$

for all $\alpha\in\mathbb{N}_0^n$ and $x\in U_1$.

If all the coefficients $c_1(x),\dots,c_N(x)$ are constant on a nonempty open subset U_2 of U_1, then

$$u(x)=\sum_{\nu=1}^{N+1} c_\nu\, u_{i_\nu}(x)$$

is an element of $\mathcal{S}_{\ker A}(U)$ which vanishes for all $x\in U_2$. By the unique continuation property, we conclude that $u\equiv 0$ in U. Since the system $\{u_i\}_{i\in I}$ is linearly independent in $\mathcal{S}_{\ker A}(U)$, we see that $c_\nu=0$ for all $\nu=1,\dots,N+1$. This leads to $-1=0$, a contradiction.

Assume that there is no nonempty open set $U_2\subset U_1$ with the property that all the coefficients $c_1(x),\dots,c_N(x)$ are constant on U_2. Then (12.1) immediately implies that

$$\partial^{\alpha+e_j}u_{i_{N+1}}(x)=\sum_{\nu=1}^{N}\Big(\partial_j c_\nu(x)\,\partial^\alpha u_{i_\nu}(x)+c_\nu(x)\,\partial^{\alpha+e_j}u_{i_\nu}(x)\Big)$$

whenever $\alpha\in\mathbb{N}_0^n$, $j=1,\dots,n$ and $x\in U_1$. Once again applying (12.1) we deduce that

$$\sum_{\nu=1}^{N}\partial_j c_\nu(x)\, j^\infty(u_{i_\nu})(x,z)=0$$

for all $j=1,\dots,n$ and $x\in U_1$. On choosing an arbitrary function $c\in\mathcal{E}(U)$ with compact support in U we get

$$\sum_{\nu=1}^{N} c(x)\,\partial_j c_\nu(x)\, j^\infty(u_{i_\nu})(x,z)=0$$

on all of U. The inductive assumption yields $c\,\partial_j c_\nu\equiv 0$ in U for all $1\le j\le n$ and $1\le\nu\le N$. In particular, $\partial_j c_\nu\equiv 0$ on the support of c for all $1\le j\le n$ and $1\le\nu\le N$. This contradicts our assumption that the functions $c_1(x),\dots,c_N(x)$ are not all constant on any nonempty open set $U_2\subset U_1$. We see that this is impossible, which completes the proof. $\square$

If a sufficiently regular differential operator A satisfies the uniqueness condition for the Cauchy problem in the small on U and the rank of $\mathcal{R}^\infty$ is finite over U, then Lemma 12.1 implies that the dimension of $\mathcal{S}_{\ker A}(U)$ is finite, too, and it does not exceed the rank of $\mathcal{R}^\infty$. Therefore, if the dimension of the space $\mathcal{S}_{\ker A}(U)$ is equal to the rank of $\mathcal{R}^\infty$ then $\mathcal{E}(U,\mathcal{R}^\infty)$ is generated by $j^\infty(\mathcal{S}_{\ker A}(U))$ over the ring $\mathcal{E}(U)$.

Theorem 12.2. *Suppose A is a sufficiently regular differential operator of type $E \to F$ on $\mathcal{X}$ satisfying the uniqueness condition for the Cauchy problem in the small. Let U be a star-shaped domain in $\mathcal{X}$, such that $\mathcal{E}(U,\mathcal{R}^\infty)$ is generated by $j^\infty(\mathcal{S}_{\ker A}(U))$ over the ring $\mathcal{E}(U)$ Then for each $f \in \mathcal{E}(U,F)$ satisfying $Bf = 0$ in U there is $u \in \mathcal{E}(U,E)$ with $Au = f$.*

Another way of stating this theorem is to say that a C^∞ Poincaré lemma holds for overdetermined systems "reducible" to Frobenius systems.

Proof. By the axiom of choice, we can choose a Hamel basis $\{u_i\}_{i\in I}$ in $\mathcal{S}_{\ker A}(U)$. Of course, if the dimension of $\mathcal{S}_{\ker A}(U)$ is finite, we need not use any axiom of choice. Fix such a basis.

By Lemma 12.1, the system $\{j^\infty(u_i)\}_{i\in I}$ is a basis in $\mathcal{E}(U,\mathcal{R}^\infty)$ over the ring $\mathcal{E}(U)$. Hence, for each $f \in \Omega^q(U,\mathcal{R}^\infty)$ there are unique indices $i_1,\dots,i_N \in I$, the number N depending on f, and differential forms $c_1,\dots,c_N \in \Omega^q(U)$, such that

$$f(x,z) = \sum_{\nu=1}^{N} c_\nu(x)\, j^\infty(u_{i_\nu})(x,z).$$

Once again using Lemma 12.1 we deduce that

$$\mathfrak{d}^q f(x,z) = \sum_{\nu=1}^{N} (dc_\nu)(x)\, j^\infty(u_{i_\nu})(x,z),$$

and $\mathfrak{d}^q f = 0$ if and only if $dc_\nu \equiv 0$ in U for all $\nu = 1,\dots,N$.

As in Section 11 we can now define a "regularisation" of the Buttin homotopy operator by $\tilde{h}^0 f = 0$ and

$$\tilde{h}^q f = \sum_{\nu=1}^{N} (h^q c_\nu)\, j^\infty(u_{i_\nu}),$$

if $q > 0$. By construction, $\tilde{h}^q$ maps $\Omega^q(U,\mathcal{R}^\infty)$ to $\Omega^{q-1}(U,\mathcal{R}^\infty)$. From the equality $\mathfrak{d}(c\, j^\infty f) = dc\, j^\infty f$ we see that

$$\tilde{h}^{q+1}\mathfrak{d}^q f + \mathfrak{d}^{q-1}\tilde{h}^q f = f$$

for $q \geq 1$. It follows that the limit complex of Spencer is exact over U. Finally, Theorem 9.4 yields that the complex

$$\mathcal{E}(U,E) \xrightarrow{A} \mathcal{E}(U,F) \xrightarrow{B} \dots$$

is exact, as desired. □

Of course, we can do the same at the level of sheaves. The overdetermined systems "reducible" to Frobenius systems need not formally have null-space of finite dimension over U. However, we know no example where the dimension is infinite.

It is worth pointing out that even for the Cauchy-Riemann operator $A = \bar{\partial}$ in $\mathbb{C}^n$, $n > 1$, the space $\mathcal{E}(U,\mathcal{R}^\infty)$ can not be generated by $j^\infty(\mathcal{S}_{\ker A}(U))$ over the ring $\mathcal{E}(U)$, if U is not pseudoconvex. This follows from the well-known fact that if the Dolbeault complex is exact at every positive step over an open set $U \subset \mathbb{C}^n$ then U is pseudoconvex.

As is observed in [But67, p. 237], it is sufficient to "regularise" the homotopy operator only on the module generated by $j^{\infty}(\mathcal{S}_{\ker A}(U))$, at least if U is pseudoconvex.

The following conjecture is very probable but we have not been able to establish it.

Theorem 12.3. *Let A be sufficiently regular and possess the unique continuation property.. Then the compatibility complex for A is exact at step $q \geq 1$ if and only if there exists $s \geq s_0$ (s_0 being the number from Quillen's Theorem 5.6), such that every $f \in \Omega^{q+1}(U, \mathcal{R}^s)$ satisfying $\mathfrak{d}^s f = 0$ in U possesses a prolongation $\tilde{f} \in \Omega^{q+1}(U, \mathcal{R}^{\infty})$ which fulfills $\mathfrak{d}\tilde{f} = 0$ in U and belongs to the linear span of $j^{\infty}(\mathcal{S}_{\ker A}(U))$ over $\Omega^{q+1}(U)$.*

Acknowledgements The first author gratefully acknowledges the financial support of the Alexander-von-Humboldt Foundation.

References

[Bjo93] Bjork, J.-E., *Analytic D -Modules and Applications*, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[But67] Buttin, C., *Existence of homotopy operator for Spencer's sequence in analytic case*, Pacific J. Math. **21** (1967), 219–240.

[Fed96] Fedosov, B., *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin, 1996.

[Gol67] Goldschmidt, H., *Existence theorems for analytic linear partial differential equations*, Ann. Math. **86** (1967), no. 2, 246–270.

[Gui68] Guillemin, V., *Some algebraic results concerning the characteristics of overdetermined partial differential equations*, Amer. J. Math. **90** (1968), 270–284.

[Har64] Hartman, P., *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[Kak99] Kakié, Kunio, *Existence of smooth solutions of overdetermined elliptic differential equations in two independent variables*, Commentarii Mathematiki Univerrsitatis Sancti Pauli **48** (1999), no. 2, 181–209.

[Kas75] Kashiwara, M., *On the maximally overdetermined systems of linear differential equations, I*, Publ. Res. Inst. Math. Sci. **10** (1974/75), 563–579.

[Kas78] Kashiwara, M., *On the holonomic systems of linear differential equations, II*, Invent. Math. **49** (1978), no. 2, 121–135.

[Lew57] Lewy, H., *An example of a smooth linear differential equations without solutions*, Ann. Math. **66** (1957), no. 2, 155–158.

[ML63] Mac Lane, S., *Homology*, Springer-Verlag, Berlin er al., 1963.

[Mal66] Malgrange, B., *Cohomologie de Spencer (d'après Quillen)*, Publ. du Seminair de Math. d'Orsay, 1966.

[Mal04] Malgrange, B., *Systèmes différentiels involutifs*, Prépublication de l'Institut Fourier **636** (2004), 111 pp.

[Miz61] Mizohata, S., *Solutions nulles et solutions non analitiques*, J. Math. Kyoto Univ. **1** (1961/62), 271–302.

[Pom78] Pommaret, J., *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach Sci. Publ., New York et al., 1978.

[Qui64] Quillen, D. C., *Formal Properties of Overdetermined Systems of Partial Differential Equations*, PhD Thesis, Harvard Univ., Harvard, 1964.

[Sam81] Samborskii, S. N., *Boundary value problems for overdetermined systems of equations with partial derivatives*, Preprint N 81.48, Inst. of Math., Kiev, 1981, 46 pp.

[Spe69] Spencer, D. C., *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc. **75** (1969), no. 2, 179–239

[Tar95] Tarkhanov, N., *Complexes of Differential Operators*, Kluwer Academic Publishers, Dordrecht, NL, 1995.

(Alexander Shlapunov) Siberian Federal University, Institute of Mathematics, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia

E-mail address: `shlapuno@lan.krasu.ru`

(Nikolai Tarkhanov) Institute of Mathematics, University of Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany

E-mail address: `tarkhanov@math.uni-potsdam.de`