On Completeness of Root Functions of Sturm-Liouville Problems with Discontinuous Boundary Operators

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Abstract

We consider a Sturm-Liouville boundary value problem in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$. By this is meant that the differential equation is given by a second order elliptic operator of divergent form in $\mathcal{D}$ and the boundary conditions are of Robin type on $\partial\mathcal{D}$. The first order term of the boundary operator is the oblique derivative whose coefficients bear discontinuities of the first kind. Applying the method of weak perturbation of compact selfadjoint operators and the method of rays of minimal growth, we prove the completeness of root functions related to the boundary value problem in Lebesgue and Sobolev spaces of various types.

Keywords: Sturm-Liouville problem, discontinuous Robin condition, root function, Lipschitz domain, non-coercive problem
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Introduction

The Hilbert space methods take considerable part in the modern theory of partial differential equations. In particular, the spectral theorem for compact selfadjoint operators attributed to Hilbert and Schmidt allows one to look for solutions of boundary value problems for formally selfadjoint operators in the form of expansions over eigenfunctions of the operator.

Non-selfadjoint compact operators fail to have eigenvectors in general. Keldysh [Kel51] (see also [Kel71] and [GK69, Ch. 5, §8] for more details) elaborated expansions over root functions for weak perturbations of compact selfadjoint operators. In particular, he applied successfully the theorem on the completeness of root functions to the Dirichlet problem for second order elliptic operators in divergent form.

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The problem of completeness of the system of eigen- and associated functions of boundary value problems for elliptic operators in domains with smooth boundary was studied in many articles (see for instance [Bro53], [Bro59a], [Bro59b], [Agm62], [Kon99]). In a series of papers [Agr94a], [Agr94b], [Agr08], [Agr11b], [Agr11c], including two surveys [Agr02] and [Agr11a], Agranovich proved the completeness of root functions for a wide class of boundary value problems for second order elliptic equations with boundary conditions of the Dirichlet, Neumann and Zaremba type in standard Sobolev spaces over domains with Lipschitz boundary. Note that the class of Lipschitz surfaces does not include surfaces with general conical points, as they are introduced in analysis on singular spaces.

Root functions of general elliptic boundary value problems in weighted Sobolev spaces over domains with conic and edge type singularities on the boundary were studied in [EKS01] and [Tar06]. These papers used estimates of the resolvent of compact operators and the so-called rays of minimal growth. In order to realize fully to what extent the completeness criteria of [EKS01] and [Tar06] are efficient, we dwell on the concept of ellipticity on a compact manifold with smooth edges on the boundary. Such a singular space $\mathcal{X}$ has three smooth strata, more precisely, the interior part $\mathcal{X}_0$ of $\mathcal{X}$, the smooth part $\mathcal{X}_1$ of the boundary and the edge $\mathcal{X}_2$ which is assumed to be a compact closed manifold. Pseudodifferential operators on $\mathcal{X}$ are (3×3)-matrices $\mathcal{A}$ whose entries $A_{i,j}$ are operators mapping functions on $\mathcal{X}_j$ to functions on $\mathcal{X}_i$. To each operator $\mathcal{A}$ one assigns a principal symbol $\sigma(\mathcal{A}) := (\sigma_0(\mathcal{A}), \sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}))$ in such a way that $\sigma(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is compact, and $\sigma(\mathcal{B}\mathcal{A}) = \sigma(\mathcal{B})\,\sigma(\mathcal{A})$ for all operators $\mathcal{A}$ and $\mathcal{B}$ whose composition is well defined. The components $\sigma_i(\mathcal{A})$ of the principal symbol are functions on the cotangent bundles of $\mathcal{X}_i$ with values in operator spaces. They are smooth away from zero sections of the bundles and bear certain twisted homogeneity as operator families. An operator $\mathcal{A}$ is called elliptic if its principal symbol is invertible away from the zero sections of cotangent bundles. The invertibility of $\sigma_0(\mathcal{A})$ just amounts to the ellipticity of $\mathcal{A}$ in the interior of $\mathcal{X}$. The invertibility of $\sigma_1(\mathcal{A})$ is equivalent to the Shapiro-Lopatinskii condition on the smooth part of $\partial\mathcal{X}$. The invertibility of $\sigma_2(\mathcal{A})$ constitutes the most difficult problem, for this operator family is considered in weighted Sobolev spaces on an infinite cone. An operator $\mathcal{A}$ proves to be Fredholm if and only if it is elliptic. However, from what has been said it follows that there is no efficient criteria of ellipticity on compact manifolds with edges on the boundary. In general these techniques allow one to derive at most the following result. Consider a classical boundary value problem on $\mathcal{X}$ satisfying the Shapiro-Lopatinskii condition away from the edge $\mathcal{X}_2$. It is actually given by a column of operators $A_{i,0}$ with $i = 0, 1$, where $A_{0,0}$ is an elliptic differential operator in $\mathcal{X}_0$ and $A_{1,0}$ a differential operator near $\mathcal{X}_1$ followed by restriction to $\mathcal{X}_1$. We complete the column to a (2×2)-matrix A by setting $A_{0,1} = 0$ and $A_{1,1} = 0$. The Shapiro-Lopatinskii condition implies that $\sigma_2(A)(y, \eta)$ is a family of Fredholm operators on the unit sphere in $T^*\mathcal{X}_2$. Hence we can set $\sigma_2(A)(y, \eta)$ in the frame of a (3×3)-matrix $a(y, \eta)$ on the unit sphere of $T^*\mathcal{X}_2$ which is moreover invertible. A distinct quantisation procedure leads then immediately to a Fredholm operator of the type

$$\begin{pmatrix} A_{0,0} & A_{0,2} \\ A_{1,0} & A_{1,2} \\ A_{2,0} & A_{2,2} \end{pmatrix} : \begin{array}{c} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_1}) \end{array} \to \begin{array}{c} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\partial\mathcal{X}, \mathbb{C}^m) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_2}) \end{array}, \tag{0.1}$$

where l_1 and l_2 are non-negative integers. However, the Fredholm property of (0.1) elucidates by no means the original problem

$$\begin{cases} A_{0,0}u &= f \quad \text{in} \quad \mathcal{X}_0, \\ A_{1,0}u &= u_0 \quad \text{on} \quad \mathcal{X}_1, \end{cases}$$

unless $\mathcal{X}_2$ is of dimension 0. Thus, operator-valued symbols make the condition of ellipticity ineffective.

In the present paper we study the completeness of root functions for the Sturm-Liouville boundary value problems for second order elliptic operators in divergent form with Robin type boundary conditions containing oblique derivative with discontinuous coefficients. The discontinuities are of first kind along a smooth hypersurface on the boundary of $\mathcal{D}$. This hypersurface can be thought of as edge on the boundary, hence we are within the framework of analysis on compact manifolds with edges of codimension 1 on the boundary. The theory of [Tar06] applies in this situation provided that one is able to establish the invertibility of the edge symbol, and this is not an easy task.

Precisely, our contribution consists in considering non-coercive forms. Indeed, a Hermitian form associated with a second order elliptic formally selfadjoint operator A is usually constructed through a factorization $A = C^*C$ with an overdetermined elliptic first order operator C and its formal adjoint C^*. According to [SKK73], microlocally any first order operator C with complex-valued coefficients can be presented via the Lewy operator or the gradient operator or the multidimensional Cauchy-Riemann operator. The Lewy-type operators go beyond elliptic theory, the holonomic operators like the gradient lead to coercive mixed problems related to A, and the Cauchy-Riemann type operators generate non-coercive boundary conditions. Thus, it is not fortuitous that non-coercive boundary value problems for elliptic differential operators attract attention of mathematicians since the middle of the 20 th century (see for instance [ADN59], [KN65]). One of the typical problems of this type is the famous $\bar{\partial}$-Neumann problem for the Dolbeault complex, and its boundary conditions involve exactly the multidimensional Cauchy-Riemann operator (see [Koh79]). The investigation of the problem resulted in the discovery of the subellipticity phenomenon which greatly influenced the development of the theory of partial differential equations (cf. [Hör66]). To the best of our knowledge, there are no advanced results on the completeness of root functions related to non-coercive problems. However, the use of non-coercive forms enlarges essentially the class of those boundary conditions for which the root functions of corresponding mixed problems are dense in L^2. The enlargement allows one to perturb the boundary conditions by diverse tangential vector fields. In general, we lose on regularity of solutions, however, this gap is well motivated by the nature of problems.

1. Weak perturbations of compact selfadjoint operators

Let H be a separable (complex) Hilbert space and $A : H \to H$ a linear operator. As usual, $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A if there is a non-zero element $u \in H$, such that $(A - \lambda I)u = 0$, where I is the identity operator in H. The element u is called an eigenvector of A corresponding to the eigenvalue λ. When supplemented with the zero element, all eigenvectors corresponding to an eigenvalue λ form a vector subspace $E(\lambda)$ in H. It is called an eigenspace of A corresponding to λ, and the dimension of $E(\lambda)$ is called the (geometric) multiplicity of λ. The famous spectral theorem of Hilbert and Schmidt asserts that the system of eigenvectors of a compact selfadjoint operator in H is complete.

Theorem 1.1. *Let $A : H \to H$ be compact and selfadjoint. Then all eigenvalues of A are real, each non-zero eigenvalue has finite multiplicity, and the system of all eigenvalues counted with their multiplicities is countable and has the only accumulation point $\lambda = 0$. Moreover, there is an orthonormal basis in H consisting of eigenvectors of A.*

As already mentioned, a non-selfadjoint compact operator might have no eigenvalues. However, each non-zero eigenvalue (if exists) is of finite multiplicity, see for instance [DS63]. Similarly to the Jordan normal form of a linear operator on a finite-dimensional vector space one uses the more general concept of root functions of operators.

More precisely, an element $u \in H$ is called a root vector of A corresponding to an eigenvalue $\lambda \in \mathbb{C}$ if $(A-\lambda I)^m u = 0$ for some natural number m. The set of all root vectors corresponding to an eigenvalue λ form a vector subspace in H whose dimension is called the (algebraic) multiplicity of λ.

If the linear span of the set of all root elements is dense in H one says that the root elements of A are complete in H. Aside from selfadjoint operators, the question arises under what conditions on a compact operator A the system of its root elements is complete.

If the dimension of H is finite then the completeness is equivalent to the possibility of reducing the matrix A to the Jordan normal form. Of course, this is always the case for linear operators in complex vector spaces, see for instance [VdW67, § 88].

In order to formulate the simplest completeness result for Hilbert spaces we need the definition of the order of a compact operator A. Since $A : H \to H$ is compact, the operator A^*A is compact, selfadjoint and non-negative. Hence it follows that A^*A possesses a unique non-negative selfadjoint compact square root $(A^*A)^{1/2}$ often denoted by $|A|$. By Theorem 1.1 the operator $|A|$ has countable system of non-negative eigenvalues $s_\nu(A)$ which are called the s-numbers of A. It is clear that if A is selfadjoint then $s_\nu = |\lambda_\nu|$, where $\{\lambda_\nu\}$ is the system of eigenvalues of A.

Definition 1.2. The operator A is said to belong to the Schatten class $\mathfrak{S}_p$, with $0 < p < \infty$, if

$$\sum_\nu |s_\nu(A)|^p < \infty.$$

Note that $\mathfrak{S}_2$ is the set of all Hilbert-Schmidt operators while $\mathfrak{S}_1$ is the ideal of all trace class operators.

The following lemma will be very useful in the sequel; it is taken from [DS63] (see also [GK69, Ch. 2, § 2]).

Lemma 1.3. *Let A be a compact operator of class $\mathfrak{S}_p$, with $0 < p < \infty$, in a Hilbert space H, and B be a bounded operator in H. Then the compositions BA and AB belong to $\mathfrak{S}_p$.*

After M.V. Keldysh a compact operator A is said to be of finite order if it belongs to a Schatten class $\mathfrak{S}_p$. The infinum of such numbers p is called the order of A. The following result is usually referred to as theorem on weak perturbations of compact selfadjoint operators. It was first proved in [Kel51], see also [Kel71]. Here we present its formulation from [GK69, Ch. 5, § 8].

Theorem 1.4. *Let A_0 be a compact selfadjoint operator of finite order in H. If δA is a compact operator and the operator $A_0(I + \delta A)$ is injective, then the system of root elements of $A_0(I + \delta A)$ is complete in H and, for any $\varepsilon > 0$, all eigenvalues of $A_0(I + \delta A)$ (except for a finite number) belong to the angles $|\arg \lambda| < \varepsilon$ and $|\arg \lambda - \pi| < \varepsilon$. Moreover,*

1) If A_0 has only a finite number of negative eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda - \pi| < \varepsilon$.

2) If A_0 has only a finite number of positive eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda| < \varepsilon$.

As is easy to see, both operators $A_0(I + \delta A)$ and A_0 are in fact injective under the hypothesis of Theorem 1.4.

2. The Sturm-Liouville problem

By a Sturm-Liouville problem in a domain in $\mathbb{R}^n$ we mean any boundary value problem for solutions of second order elliptic partial differential equation with Robin-type boundary condition. The coefficients of the Robin boundary condition are allowed to have discontinuities of the first kind, and so mixed boundary conditions are included as well.

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, i.e., the surface $\partial\mathcal{D}$ is locally the graph of a Lipschitz function. In particular, the boundary $\partial\mathcal{D}$ possesses a tangent hyperplane almost everywhere.

We consider complex-valued functions defined in the domain $\mathcal{D}$ and its closure $\overline{\mathcal{D}}$. For $1 \leq q \leq \infty$, we write $L^q(\mathcal{D})$ for the space of all measurable functions u in $\mathcal{D}$, such that the integral of $|u|^q$ over $\mathcal{D}$ is finite. Assume s is a non-negative integer. For functions $u \in C^\infty(\overline{\mathcal{D}})$ we introduce the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\int_{\mathcal{D}} \sum_{|\alpha|\leq s} |\partial^\alpha u|^2 \, dx \Big)^{1/2},$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots \alpha_n)$, and $\partial_j = \partial/\partial x_j$. The completion of the space $C^\infty(\overline{\mathcal{D}})$ with respect to this norm is the Banach space $H^s(\mathcal{D})$. It is convenient to set $H^0(\mathcal{D}) := L^2(\mathcal{D})$. Then $H^s(\mathcal{D})$ is a Hilbert space with scalar product

$$(u, v)_{H^s(\mathcal{D})} = \int_{\mathcal{D}} \sum_{|\alpha|\leq s} \partial^\alpha u \overline{\partial^\alpha v} \, dx,$$

$u, v \in H^s(\mathcal{D})$.

For a positive non-integer s, we introduce the Sobolev-Slobodetskii space $H^s(\mathcal{D})$ in the same way as the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\|u\|^2_{H^{[s]}(\mathcal{D})} + \iint_{\mathcal{D}\times\mathcal{D}} \sum_{|\alpha|=[s]} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} \, dxdy \Big)^{1/2},$$

where $[s]$ is the integer part of s.

We consider a second order partial differential operator A in the domain $\mathcal{D}$ of divergence form

$$A(x, \partial)u = -\sum_{i,j=1}^{n} \partial_i (a_{i,j}(x)\partial_j u) + \sum_{j=1}^{n} a_j(x)\partial_j u + a_0(x)u,$$

the coefficients $a_{i,j}$, a_j and a_0 being bounded functions in $\mathcal{D}$. We emphasize that the coefficients are allowed to take on complex values, too.

Let $\vartheta(x) = (\vartheta_1(x), \dots, \vartheta_n(x))$ be a vector field on the surface $\partial\mathcal{D}$ taking on its values in $\mathbb{C}^n$. Denote by ∂_ϑ the oblique derivative

$$\partial_\vartheta = \sum_{j=1}^{n} \vartheta_j(x)\partial_j \tag{2.1}$$

and introduce a first order boundary operator $B(x, \partial) = \partial_\vartheta + b_0(x)$. The coefficients $\vartheta_1(x), \dots, \vartheta_n(x)$ and $b_0(x)$ are assumed to be bounded measurable functions on $\partial\mathcal{D}$. We allow the vector $\vartheta(x)$ to vanish on an open connected subset S of $\partial\mathcal{D}$ with piecewise smooth boundary ∂S. In this case we assume that $b_0(x)$ does not vanish on S.

Consider the following boundary value problem with Robin-type condition on the surface $\partial\mathcal{D}$. Given a distribution f in $\mathcal{D}$, find a distribution u in $\mathcal{D}$ which satisfies

$$\left\{ \begin{array}{rcll} Au & = & f & \text{in} \quad \mathcal{D}, \\ Bu & = & 0 & \text{on} \quad \partial\mathcal{D}. \end{array} \right. \tag{2.2}$$

In order to get asymptotic results, it is necessary to put some restrictions on the operators A and B. Suppose that, for each $x \in \mathcal{D}$, the matrix

$$\left(a_{i,j}(x) \right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

is non-negative semi-definite, i.e. it is Hermitian and

$$\sum_{i,j=1}^{n} a_{i,j}(x) \overline{w_i} w_j \geq 0 \tag{2.3}$$

for every choice of complex numbers $w_1, \dots, w_n$. Such operators A are elliptic although in general elliptic operators may not be non-negative. We make a stronger condition on the ellipticity, namely, there is a constant $m > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geq m \, |\xi|^2 \tag{2.4}$$

for all $(x, \xi) \in \mathcal{D} \times (\mathbb{R}^n \setminus \{0\})$. Estimate (2.4) is nothing but the statement that the operator A is strongly elliptic. Note that conditions (2.3), (2.4) are weaker than the coercivity, i.e., the existence of a constant m such that

$$\sum_{i,j=1}^{n} a_{i,j}(x) \, \overline{\partial_i u} \partial_j u \geq m \sum_{j=1}^{n} |\partial_j u|^2 \tag{2.5}$$

for all $u \in C^\infty(\overline{\mathcal{D}})$, because u may take on complex values. In §5 we indicate a typical example where both (2.3) and (2.4) are fulfilled while (2.5) fails to be true.

If the coefficients $a_{i,j}$ are continuous up to the boundary of $\mathcal{D}$, we consider the complex vector field $c(x)$ on $\partial\mathcal{D}$ with components

$$c_j(x) = \sum_{i=1}^{n} a_{i,j}(x) \nu_i(x),$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector of $\partial\mathcal{D}$ at $x \in \partial\mathcal{D}$. From condition (2.4) it follows that there are a complex-valued function $b_1(x) \in L^\infty(\partial\mathcal{D})$ and a tangential vector field $t(x)$ on $\partial\mathcal{D}$ whose components belong to $L^\infty(\partial\mathcal{D})$, such that

$$B(x, \partial) = b_1(x) \partial_c + \partial_t + b_0(x), \tag{2.6}$$

where ∂_c is the oblique derivative related to the field c (see (2.1)). By assumption, both b_1 and t vanish on S. Concerning the behavior of b_1 in the complement of S we require that $b_1(x) \neq 0$ for almost all $x \in \partial\mathcal{D} \setminus S$ and $1/b_1 \in L^1(\partial\mathcal{D} \setminus S)$. Note that in this way the Shapiro-Lopatinskii condition may be violated on the smooth part of $\partial\mathcal{D} \setminus S$ unless the coefficients $a_{i,j}$ are real-valued functions.

However we need not assume the continuity of the coefficients $a_{i,j}$ up to the boundary of D in the course of the paper, because for our purposes it is more natural to understand the boundary conditions in a weak sense. Let us be more precise.

Denote by $H^1(\mathcal{D}, S)$ the subspace of $H^1(\mathcal{D})$ consisting of those functions whose restriction to the boundary vanishes on S. This space is Hilbert under the induced norm. Furthermore, let $C^1_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus S)$ stand for the set of all smooth functions on $\overline{D}$ vanishing in a neighborhood of S. It is easily seen that $C^1_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus S)$ is dense in $H^1(\mathcal{D}, S)$. Since on S the boundary operator reduces to $B = b_0$ and $b_0(x) \neq 0$ for $x \in S$, the functions of $H^1(\mathcal{D})$ satisfying $Bu = 0$ on $\partial\mathcal{D}$ belong to $H^1(\mathcal{D}, S)$.

As we want to study perturbation of selfadjoint operators we split both a_0 and b_0 into two parts

$$\begin{array}{rcl} a_0 & = & a_{0,0} + \delta a_0, \\ b_0 & = & b_{0,0} + \delta b_0, \end{array}$$

where $a_{0,0}$ is a non-negative bounded function in $\mathcal{D}$ and $b_{0,0}$ a bounded function on $\partial\mathcal{D}$ satisfying $b_{0,0}/b_1 \geq 0$. If the functional

$$\|u\|_{SL} = \Big(\int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \overline{\partial_i u} \partial_j u \, dx + \| \sqrt{a_{0,0}} u\|^2_{L^2(\mathcal{D})} + \| \sqrt{b_{0,0} b_1^{-1}} u\|^2_{L^2(\partial\mathcal{D}\setminus S)} \Big)^{1/2}$$

defines a norm on $H^1(\mathcal{D}, S)$, we denote by $H_{SL}(\mathcal{D})$ the completion of $H^1(\mathcal{D}, S)$ with respect to this norm. Then $H_{SL}(\mathcal{D})$ is actually a Hilbert space with scalar product

$$(u, v)_{SL} = \int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \partial_j u \overline{\partial_i v} \, dx + (a_{0,0} u, v)_{L^2(\mathcal{D})} + (b_{0,0} b_1^{-1} u, v)_{L^2(\partial\mathcal{D}\setminus S)}.$$

From now on we assume that the space $H_{SL}(\mathcal{D})$ is continuously embedded into the Lebesgue space $L^2(\mathcal{D})$, i.e.,

$$\|u\|_{L^2(\mathcal{D})} \leq c \, \|u\|_{SL} \tag{2.7}$$

for all $u \in H_{SL}(\mathcal{D})$, where c is a constant independent of u. This condition is not particularly restrictive. Of course, it is true if there is a positive constant c_1, such that

$$a_{0,0} \geq c_1 \tag{2.8}$$

in $\mathcal{D}$.

In any case, if the coefficients $a_{i,j}$ are continuous in $\mathcal{D} \cup S$ then the functions of $H_{SL}(\mathcal{D})$ belong actually to $H^1_{\mathrm{loc}}(\mathcal{D} \cup S)$ because of the classical Gårding inequality for strongly elliptic equations. In particular, if $S = \partial\mathcal{D}$ then $H_{SL}(\mathcal{D})$ is continuously embedded into $H^1(\mathcal{D}, \partial\mathcal{D}) = H^1_0(\mathcal{D})$. As this situation corresponds to the well-studied Dirichlet problem, we will be mostly focus upon the case $S \neq \partial\mathcal{D}$.

Let us describe two typical embeddings for the space $H_{SL}(\mathcal{D})$, either of them implying an estimate (2.7). To this end, denote by ι the inclusion

$$H_{SL}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D}), \tag{2.9}$$

which is continuous by (2.7).

We use inclusion (2.9) to specify the dual space of $H_{SL}(\mathcal{D})$ via the pairing in $L^2(\mathcal{D})$. More precisely, let $H^-_{SL}(\mathcal{D})$ be the completion of $H^1(\mathcal{D}, S)$ with respect to the norm

$$\|u\|_{H^-_{SL}(\mathcal{D})} = \sup_{\substack{v \in H^1(\mathcal{D},S) \\ v \neq 0}} \frac{|(v, u)_{L^2(\mathcal{D})}|}{\|v\|_{SL}},$$

cf. [Sch60].

Remark 2.1. As $H^1(\mathcal{D}, S)$ is dense in $H_{SL}(\mathcal{D})$ and the norm $\|\cdot\|_{SL}$ majorizes $\|\cdot\|_{L^2(\mathcal{D})}$ we conclude that

$$\|u\|_{H^-_{SL}(\mathcal{D})} = \sup_{\substack{v \in H_{SL}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{L^2(\mathcal{D})}|}{\|v\|_{SL}}.$$

The following two lemmas are well known (see for instance [Sch60, § 3]).

Lemma 2.2. *The space $L^2(\mathcal{D})$ is continuously embedded into $H^-_{SL}(\mathcal{D})$. If inclusion (2.9) is compact then the space $L^2(\mathcal{D})$ is compactly embedded into $H^-_{SL}(\mathcal{D})$.*

Since $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$ and the norm $\|\cdot\|_{L^2(\mathcal{D})}$ majorizes the norm $\|\cdot\|_{H^-_{SL}(\mathcal{D})}$, we conclude that, under (2.7), the space $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $H^-_{SL}(\mathcal{D})$, too.

Lemma 2.3. *The Banach space $H^-_{SL}(\mathcal{D})$ is topologically isomorphic to the dual space $H_{SL}(\mathcal{D})'$ and the isomorphism is defined by the sesquilinear form*

$$(v, u) = \lim_{\nu \to \infty} (v, u_\nu)_{L^2(\mathcal{D})} \tag{2.10}$$

for $u \in H^-_{SL}(\mathcal{D})$ and $v \in H_{SL}(\mathcal{D})$ where $\{u_\nu\}$ is any sequence in $H^1(\mathcal{D}, S)$ converging to u.

That is, for every fixed $u \in H^-_{SL}(\mathcal{D})$, pairing (2.10) defines a continuous linear functional f_u on $H_{SL}(\mathcal{D})$ and, for each $f \in H_{SL}(\mathcal{D})'$, there is a unique $u \in H^-_{SL}(\mathcal{D})$ with $f(v) = f_u(v)$ for all $v \in H_{SL}(\mathcal{D})$. Moreover, the conjugate linear map $u \mapsto f_u$ is an isometry.

Note that $H_{SL}(\mathcal{D})$ is reflexive, since it is a Hilbert space. Hence it follows immediately that $(H^-_{SL}(\mathcal{D}))' = H_{SL}(\mathcal{D})$, i.e., the spaces $H_{SL}(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$ are dual to each other with respect to (2.10).

Now, for $s < 0$, the space $H^s(\mathcal{D})$ is defined to be the dual space of $H^{-s}(\mathcal{D})$ with respect to the $L^2(\mathcal{D})$-pairing, as discussed above.

As it is to be expected, the strongest embeddings of $H_{SL}(\mathcal{D})$ are reachable in the coercive case.

Lemma 2.4. *Suppose estimate (2.5) is fulfilled. Then there are continuous embeddings*

$$\begin{array}{rcl} H_{SL}(\mathcal{D}) & \hookrightarrow & H^1(\mathcal{D}, S), \\ H^{-1}(\mathcal{D}) & \hookrightarrow & H^-_{SL}(\mathcal{D}) \end{array}$$

if at least one of the following conditions holds:

1) S is not empty; 2) $\displaystyle\int_{\mathcal{D}} a_{0,0}(x)dx > 0$; 3) $\displaystyle\int_{\partial\mathcal{D}\setminus S} \frac{b_{0,0}(x)}{b_1(x)}\, ds > 0$.

In particular, in either of the cases inclusion (2.9) is compact, which is due to the Rellich-Kondrashov theorem.

Proof. The proof is standard, cf. for instance [Mik76, Ch. 3, § 5.6]. $\square$

The study of non-coercive boundary conditions includes essentially weaker embeddings of the space $H_{SL}(\mathcal{D})$.

Theorem 2.5. *Let the coefficients $a_{i,j}$ be C^∞ in a neighborhood of the closure of $\mathcal{D}$ and*

$$\frac{b_{0,0}}{b_1} \geq c_2 \tag{2.11}$$

at $\partial\mathcal{D} \setminus S$, with some constant $c_2 > 0$. Then the space $H_{SL}(\mathcal{D})$ is continuously embedded into $H^{1/2-\varepsilon}(\mathcal{D})$ for any $\varepsilon > 0$ if (2.8) is fulfilled or the operator A is strongly elliptic in a neighbourhood $\mathcal{X}$ of $\overline{\mathcal{D}}$ and

$$\int_{\mathcal{X}} \sum_{i,j=1}^{n} a_{i,j}\partial_j u \overline{\partial_i u}\, dx \geq m\, \|u\|^2_{L^2(\mathcal{X})} \tag{2.12}$$

for all $u \in C^\infty_{\mathrm{comp}}(\mathcal{X})$, with $m > 0$ a constant independent of u.

In particular, under the hypotheses of Theorem 2.5, the inclusion ι is compact.

Proof. Without loss of generality we may assume that $\mathcal{X}$ is a domain with smooth boundary and the coefficients $a_{i,j}$ are smooth in $\overline{\mathcal{X}}$. As the operator

$$A_0 = -\sum_{i,j=1}^{n} \partial_i(a_{i,j}\partial_j)$$

is strongly elliptic on $\mathcal{X}$, the classical Gårding inequality yields the existence of a Hodge parametrix $\mathcal{G}$ for the Dirichlet problem related to A_0 in $\mathcal{X}$ (see for instance [LU73] or [Sch60]). To formulate this more precisely, we define $\tilde{H}^{-1}(\mathcal{X})$ to be the dual space of $H^1_0(\mathcal{X})$ with respect to the $L^2(\mathcal{X})$-pairing, as discussed above. Clearly, $H^{-1}(\mathcal{X})$ is continuously embedded into $\tilde{H}^{-1}(\mathcal{X})$. As usual, the operator A_0 is given the domain $H^1_0(\mathcal{X})$ to map it to $\tilde{H}^{-1}(\mathcal{X})$. Then there are bounded linear operators

$$\begin{array}{rccc} \mathcal{G}: & \tilde{H}^{-1}(\mathcal{X}) & \to & H^1_0(\mathcal{X}), \\ \mathcal{H}: & \tilde{H}^{-1}(\mathcal{X}) & \to & \mathcal{H}(\mathcal{X}) \end{array}$$

satisfying

$$\begin{array}{rcl} \mathcal{G}A_0 & = & I - \mathcal{H}, \\ A_0\mathcal{G} & = & I - \mathcal{H} \end{array}$$

on $H^1_0(\mathcal{X})$ and $\tilde{H}^{-1}(\mathcal{X})$, respectively, where $\mathcal{H}(\mathcal{X}) \subset H^1_0(\mathcal{X}) \cap C^\infty(\overline{\mathcal{X}})$ stands for the null-space of the Dirichlet problem in $\mathcal{X}$. The dimension of $\mathcal{H}(\mathcal{X})$ is finite and $\mathcal{H}$ is actually the $L^2(\mathcal{X})$-orthogonal projection onto $\mathcal{H}(\mathcal{X})$. Moreover, $\mathcal{H}$ maps $\tilde{H}^{-1}(\mathcal{X})$ continuously to $C^\infty(\overline{\mathcal{X}})$ for all $s \geq -1$.

Denote by e^+ the operator of extension by zero from $\mathcal{D}$ to $\mathcal{X}$, and by r^+ the restriction from $\mathcal{X}$ to the domain $\mathcal{D}$. Obviously, e^+ is a bounded linear operator from $L^2(\mathcal{D})$ to $L^2(\mathcal{X})$ and r^+ a bounded linear operator from $H^s(\mathcal{X})$ to $H^s(\mathcal{D})$, for any $s \in \mathbb{R}$. As the coefficients $a_{i,j}$ are smooth in the closure of $\mathcal{X}$, we deduce that any solution $u \in H^1_0(\mathcal{X})$ of the Dirichlet problem with $A_0 u \in L^2(\mathcal{X})$ belongs actually to $H^2(\mathcal{X})$. Hence, on setting $\mathcal{G}_{\mathcal{D}} = r^+\mathcal{G}e^+$ and $\mathcal{H}_{\mathcal{D}} = r^+\mathcal{H}e^+$ we get bounded operators

$$\begin{array}{rccc} \mathcal{G}_{\mathcal{D}}: & L^2(\mathcal{D}) & \to & H^2(\mathcal{D}), \\ \mathcal{H}_{\mathcal{D}}: & L^2(\mathcal{D}) & \to & C^\infty(\overline{\mathcal{D}}). \end{array}$$

For $s \geq 0$, any element $u \in H^{-s}(\mathcal{D})$ extends to an element $U \in H^{-s}(\mathcal{X})$ via

$$\langle U, v\rangle_{\mathcal{X}} = \langle u, r^+ v\rangle_{\mathcal{D}}$$

for all $v \in H^s(\mathcal{X})$. Here, we use $\langle \cdot, \cdot\rangle_{\mathcal{X}}$ to designate the pairing of dual spaces of distributions in $\mathcal{X}$. It will cause no confusion if we write e^+u for this extension U, for the support of U is still contained in $\overline{\mathcal{D}}$. Obviously, the extension operator e^+ defined in this way maps $H^{-s}(\mathcal{D})$ continuously to $H^{-s}(\mathcal{X})$ for all real $s \geq 0$. Using now the continuity properties of pseudodifferential operators on compact closed manifolds we deduce that both $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}}$ extend to bounded linear operators

$$\begin{array}{rccc} \mathcal{G}_{\mathcal{D}}: & H^{-1/2+\varepsilon}(\mathcal{D}) & \to & H^{3/2+\varepsilon}(\mathcal{D}), \\ \mathcal{H}_{\mathcal{D}}: & H^{-1/2+\varepsilon}(\mathcal{D}) & \to & C^\infty(\overline{\mathcal{D}}) \end{array}$$

for any $0 < \varepsilon \leq 1/2$. Hence, the operators

$$\begin{array}{rccc} \partial_j \mathcal{G}_{\mathcal{D}}: & H^{-1/2+\varepsilon}(\mathcal{D}) & \to & H^{1/2+\varepsilon}(\mathcal{D}), \\ \partial_c \mathcal{G}_{\mathcal{D}}: & H^{-1/2+\varepsilon}(\mathcal{D}) & \to & H^{\varepsilon}(\partial\mathcal{D}) \end{array}$$

are bounded, too, if $0 < \varepsilon \leq 1/2$, the continuity of the latter operator is a consequence of the trace theorem for Sobolev-Slobodetskii spaces. For $\varepsilon = 0$ the arguments fail because the elements of $H^{1/2}(\mathcal{D})$ need not have any traces on $\partial\mathcal{D}$.

Clearly, $\mathcal{H} \equiv 0$ if (2.12) is fulfilled. On the other hand, if condition (2.8) holds true then $H_{SL}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$. Hence the norm $\|\cdot\|_{SL}$ is not weaker than the norm $\|\cdot\|_a$ on $H^1(\mathcal{D}, S)$ defined by

$$\|u\|_a = \Big(\int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \partial_j u \overline{\partial_i u}\, dx + \|u\|^2_{L^2(\partial\mathcal{D}\setminus S)} + \|\mathcal{H} e^+ u\|^2_{L^2(\mathcal{X})} \Big)^{1/2}.$$

Pick a real number $\varepsilon > 0$. Let us show that the norm $\|\cdot\|_a$ is not weaker than the norm $\|\cdot\|_{H^{1/2-\varepsilon}(\mathcal{D})}$ on $H^1(\mathcal{D}, S)$.

As the coefficients $a_{i,j}(x)$ are smooth up to the boundary of $\mathcal{D}$, the Stokes formula yields

$$\int_{\partial\mathcal{D}} \overline{v} \partial_c u\, ds = \int_{\mathcal{D}} \sum_{i,j=1}^n \Big(a_{i,j} \overline{\partial_i v} \partial_j u + \overline{v} \partial_i (a_{i,j} \partial_j u) \Big) dx. \tag{2.13}$$

Hence it follows that

$$\begin{array}{rcl} (v, u)_{L^2(\mathcal{D})} & = & (\mathcal{H}_{\mathcal{D}} v + A_0 \mathcal{G}_{\mathcal{D}} v, u)_{L^2(\mathcal{D})} \\ & = & (\mathcal{H}_{\mathcal{D}} v, u)_{L^2(\mathcal{D})} + \displaystyle\int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \partial_j (\mathcal{G}_{\mathcal{D}} v) \overline{\partial_i u}\, dx + (\partial_c(\mathcal{G}_{\mathcal{D}} v), u)_{L^2(\partial\mathcal{D}\setminus S)} \end{array}$$

for all $u \in H^1(\mathcal{D}, S)$ and $v \in L^2(\mathcal{D})$.

Let now $v \in H^{\varepsilon-1/2}(\mathcal{D})$, where $0 < \varepsilon < 1/2$. Take a sequence $\{v_k\}$ in $C^\infty(\overline{\mathcal{D}})$ converging to v in the space $H^{\varepsilon-1/2}(\mathcal{D})$. Using the above formula and the continuity of operators $\mathcal{H}_{\mathcal{D}}$, $\partial_j \mathcal{G}_{\mathcal{D}}$ and $\partial_c \mathcal{G}_{\mathcal{D}}$, we get

$$(v, u) = \lim_{k\to+\infty} (v_k, u)_{L^2(\mathcal{D})} = (\mathcal{H}_{\mathcal{D}} v, u)_{L^2(\mathcal{D})} + \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \partial_j (\mathcal{G}_{\mathcal{D}} v) \overline{\partial_i u}\, dx + (\partial_c(\mathcal{G}_{\mathcal{D}} v), u)_{L^2(\partial\mathcal{D}\setminus S)}$$

for all $u \in H^1(\mathcal{D}, S)$. As the space $H^s(\mathcal{D})$ is reflexive for each s, it follows that

$$\begin{aligned} \|u\|_{H^{1/2-\varepsilon}(\mathcal{D})} &= \sup_{\substack{v \in H^{-1/2+\varepsilon}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)|}{\|v\|_{H^{-1/2+\varepsilon}(\mathcal{D})}} \\ &= \sup_{\substack{v \in H^{-1/2+\varepsilon}(\mathcal{D}) \\ v \neq 0}} \frac{\Big|(\mathcal{H}_{\mathcal{D}} v, u)_{L^2(\mathcal{D})} + \int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \partial_j (\mathcal{G}_{\mathcal{D}} v) \overline{\partial_i u}\, dx + (\partial_c(\mathcal{G}_{\mathcal{D}} v), u)_{L^2(\partial\mathcal{D}\setminus S)}\Big|}{\|v\|_{H^{-1/2+\varepsilon}(\mathcal{D})}} \end{aligned} \tag{2.14}$$

for all $u \in H^1(\mathcal{D}, S)$. As $\mathcal{H}$ is an $L^2(\mathcal{X})$-orthogonal projection, we obtain

$$\begin{aligned} |(\mathcal{H}_{\mathcal{D}} v, u)_{L^2(\mathcal{D})} &= |(\mathcal{H} e^+ v, e^+ u)_{L^2(\mathcal{X})}| \\ &= |(\mathcal{H}^2 e^+ v, e^+ u)_{L^2(\mathcal{X})}| \\ &\leq c_1\, \|v\|_{H^{-1/2+\varepsilon}(\mathcal{D})} \|\mathcal{H} e^+ u\|_{L^2(\mathcal{X})} \end{aligned} \tag{2.15}$$

and

$$|(\partial_c(\mathcal{G}_{\mathcal{D}} v), u)_{L^2(\partial\mathcal{D}\setminus S)}| \leq c_2\, \|v\|_{H^{-1/2+\varepsilon}(\mathcal{D})} \|u\|_{L^2(\partial\mathcal{D}\setminus S)} \tag{2.16}$$

for all $v \in H^{-1/2+\varepsilon}(\mathcal{D})$ and $u \in H^1(\mathcal{D}, S)$, with c_1 and c_2 positive constants independent of u and v. Finally, using the generalized Cauchy inequality we see that

$$\Big| \int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \partial_j (\mathcal{G}_{\mathcal{D}} v) \overline{\partial_i u}\, dx \Big| \leq c \Big(\int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \partial_j u \overline{\partial_i u}\, dx \Big)^{1/2} \|v\|_{H^{-1/2+\varepsilon}(\mathcal{D})} \tag{2.17}$$

with constant $c = \Big(\sum_{i,j=1}^{n} \sup_{x \in \mathcal{D}} |a_{i,j}(x)|\, \|\partial_i \mathcal{G}_{\mathcal{D}}\|\, \|\partial_j \mathcal{G}_{\mathcal{D}}\| \Big)^{1/2}$.

On combining the estimates (2.14), (2.15), (2.16) and (2.17) we deduce readily that there are constants $C_1 > 0$ and $C_2 > 0$, such that

$$\|u\|_{H^{1/2-\varepsilon}(\mathcal{D})} \leq C_1\, \|u\|_a \leq C_2\, \|u\|_{SL}$$

for all $u \in H^1(\mathcal{D}, S)$. This proves the continuous embedding $H_{SL}(\mathcal{D}) \hookrightarrow H^{1/2-\varepsilon}(\mathcal{D})$ with any $\varepsilon > 0$, as desired. □

Actually condition (2.12) of Theorem 2.5 is much weaker than the coercive estimate of (2.5). It is fulfilled if, for instance, the diameter of the domain $\mathcal{D}$ is sufficiently small or if the coefficients $a_{i,j}$ are real analytic in $\mathcal{X}$. We also note that for non-coercive boundary value problems the embedding described in Theorem 2.5 is rather sharp, see Remark 5.1 below.

We now proceed to study the Sturm-Liouville problem. If the coefficients $a_{i,j}$ are continuous up to the boundary of $\mathcal{D}$ then, integrating by parts with the use of (2.13), we see that

$$(Au, v)_{L^2(\mathcal{D})} = \int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j} \overline{\partial_i v} \partial_j u\, dx + \Big(b_1^{-1}(\partial_t + b_0) u, v \Big)_{L^2(\partial\mathcal{D}\setminus S)} + \Big(\sum_{j=1}^{n} a_j \partial_j u + a_0 u, v \Big)_{L^2(\mathcal{D})} \tag{2.18}$$

for all $u \in H^2(\mathcal{D})$ and $v \in H^1(\mathcal{D})$ satisfying the boundary condition of (2.2). This identity suggests to generalize the setting for the case of bounded coefficients $a_{i,j} \in L^\infty(\mathcal{D})$. Namely, from now on we assume that

$$\Big| \Big(b_1^{-1}(\partial_t + \delta b_0)u, v \Big)_{L^2(\partial\mathcal{D}\setminus S)} + \Big(\sum_{j=1}^n a_j \partial_j u + \delta a_0\, u, v \Big)_{L^2(\mathcal{D})} \Big| \leq c\, \|u\|_{SL} \|v\|_{SL} \tag{2.19}$$

for all $u, v \in H^1(\mathcal{D}, S)$, where c is a positive constant independent of u and v. In Sections 5 and 6 we provide explicit conditions for estimate (2.19) to hold.

Then, for each fixed $u \in H_{SL}(\mathcal{D})$, the sesquilinear form

$$Q(u, v) := \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \overline{\partial_i v} \partial_j u\, dx + \Big(b_1^{-1}(\partial_t + b_0)u, v \Big)_{L^2(\partial\mathcal{D}\setminus S)} + \Big(\sum_{j=1}^n a_j \partial_j u + a_0 u, v \Big)_{L^2(\mathcal{D})}$$

determines a continuous linear functional f on $H_{SL}(\mathcal{D})$ by $f(v) := \overline{Q(u, v)}$ for $v \in H_{SL}(\mathcal{D})$. By Lemma 2.3, there is a unique element in $H^-_{SL}(\mathcal{D})$, which we denote by Lu, such that

$$f(v) = (v, Lu)$$

for all $v \in H_{SL}(\mathcal{D})$. We have thus defined a linear operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$. From (2.19) it follows that L is bounded.

The bounded linear operator $L_0 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ defined in the same way via the sesquilinear form $(\cdot, \cdot)_{SL}$, i.e.,

$$(v, u)_{SL} = (v, L_0 u) \tag{2.20}$$

for all $u, v \in H_{SL}(\mathcal{D})$, corresponds to the case $a_j \equiv 0$ for all $j = 1, \ldots, n$, $a_0 = a_{0,0}$, and $t \equiv 0$, $b_0 = b_{0,0}$.

We are thus lead to a weak formulation of problem (2.2). Given $f \in H^-_{SL}(\mathcal{D})$, find $u \in H_{SL}(\mathcal{D})$, such that

$$\overline{Q(u, v)} = (v, f) \tag{2.21}$$

for all $v \in H_{SL}(\mathcal{D})$.

Thus, under conditions (2.7) and (2.19), there is no need to assume the continuity of the coefficients $a_{i,j}$ up to the boundary of $\mathcal{D}$ in order to introduce the operator L and weak formulation of problem (2.2). Now one can handle problem (2.21) by standard techniques of functional analysis, see for instance [LU73, Ch. 3, §§ 4-6]) for the coercive case.

Lemma 2.6. *Let estimate (2.7) hold true. Assume that $a_j \equiv 0$ for all $j = 1, \ldots, n$, $\delta a_0 = 0$, and $t \equiv 0$, $\delta b_0 = 0$. Then for each $f \in H^-_{SL}(\mathcal{D})$ there is a unique solution $u \in H_{SL}(\mathcal{D})$ to problem (2.21), i.e., the operator $L_0 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is continuously invertible. Moreover,*

$$\|L_0 u\|_{H^-_{SL}(\mathcal{D})} = \|u\|_{SL} \tag{2.22}$$

for all $u \in H_{SL}(\mathcal{D})$, i.e. the norms of both L_0 and its inverse L_0^{-1} just amount to 1.

Proof. The lemma follows readily from the Riesz theorem on the general form of continuous linear functionals on a Hilbert space. □

We emphasize that the space $H^-_{SL}(\mathcal{D})$ could be first introduced as the dual to $H_{SL}(\mathcal{D})$ without specifying explicit pairing. However, the duality of Lemma 2.3 provides the direct connection between the weak formulation (2.21) of the mixed problem and the integration by parts formula (2.18).

Consider the sesquilinear form on $H^-_{SL}(\mathcal{D})$ given by

$$(u, v)_{H^-_{SL}(\mathcal{D})} := (L_0^{-1} u, v)$$

for $u, v \in H^-_{SL}(\mathcal{D})$. Since

$$(L_0^{-1} u, v) = (L_0^{-1} u, L_0 L_0^{-1} v) = (L_0^{-1} u, L_0^{-1} v)_{SL} \tag{2.23}$$

for all $u, v \in H^-_{SL}(\mathcal{D})$, the last equality being due to (2.20), this form is Hermitian. Combining (2.22) and (2.23) yields

$$\sqrt{(u, u)_{H^-_{SL}(\mathcal{D})}} = \|u\|_{H^-_{SL}(\mathcal{D})}$$

for all $u \in H^-_{SL}(\mathcal{D})$. From now on we endow the space $H^-_{SL}(\mathcal{D})$ with the scalar product $(\cdot, \cdot)_{H^-_{SL}(\mathcal{D})}$.

Lemma 2.7. *Let estimates (2.7) and (2.19) be fulfilled. If moreover the constant c of (2.19) satisfies $c < 1$ then, for each $f \in H^-_{SL}(\mathcal{D})$, there exists a unique solution $u \in H_{SL}(\mathcal{D})$ to problem (2.21), i.e., the operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is continuously invertible.*

In Sections 5 and 6 we will obtain estimates for the constant c of (2.19) under reasonable assumptions. The estimates will depend upon the data S and $a_{i,j}$, a_j, b_j of the problem.

Since $C^\infty_{\mathrm{comp}}(\mathcal{D}) \hookrightarrow H_{SL}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$, the elements of $H^-_{SL}(\mathcal{D})$ are distributions in $\mathcal{D}$ and any solution to problem (2.2) satisfies $Au = f$ in $\mathcal{D}$ in the sense of distributions. Though the boundary conditions are interpreted in a weak sense, they agree with those in terms of restrictions to $\partial\mathcal{D}$ if the solution is sufficiently smooth up to the boundary, e.g. belongs to $C^1(\overline{\mathcal{D}})$. Suppose for instance that the coefficients $a_{i,j}$ are smooth in $\overline{\mathcal{D}}$ and $f \in L^2(\mathcal{D})$. Since A is elliptic, we deduce readily that $u \in H^2_{\mathrm{loc}}(\mathcal{D})$ and the equality $Au = f$ is actually satisfied almost everywhere in $\mathcal{D}$. If, in addition, $u \in H^2(\mathcal{D})$ then

$$\Big((\partial_c + b_1^{-1}(\partial_t + b_0))u, v\Big)_{L^2(\partial\mathcal{D}\setminus S)} = 0$$

for all $v \in H_{SL}(\mathcal{D})$. As any smooth function v in $\overline{\mathcal{D}}$ whose support does not meet S belongs to $H_{SL}(\mathcal{D})$, we conclude that $(b_1\partial_c + \partial_t + b_0)u = 0$ on $\partial\mathcal{D} \setminus S$. Hence, in this case $Bu = 0$ on $\partial\mathcal{D}$, for $u = 0$ and $b_1 = 0$ on S.

3. Completeness of root functions for weak perturbations

We are now in a position to study the completeness of root functions related to problem (2.21). We begin with the selfadjoint operator L_0. To this end we write ι' for the continuous embedding of $L^2(\mathcal{D})$ into $H^-_{SL}(\mathcal{D})$, as it is described by Lemma 2.2.

Lemma 3.1. *Suppose that estimate (2.7) is fulfilled and inclusion (2.9) is compact. Then the inverse L_0^{-1} of the operator given by (2.20) induces compact positive selfadjoint operators*

$$\begin{array}{rclcrcl} Q_1 & = & \iota'\iota\, L_0^{-1} & : & H^-_{SL}(\mathcal{D}) & \to & H^-_{SL}(\mathcal{D}), \\ Q_2 & = & \iota\, L_0^{-1}\iota' & : & L^2(\mathcal{D}) & \to & L^2(\mathcal{D}), \\ Q_3 & = & L_0^{-1}\iota'\iota & : & H_{SL}(\mathcal{D}) & \to & H_{SL}(\mathcal{D}) \end{array}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are positive and there are orthonormal bases in $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$ consisting of the eigenvectors.

Proof. The proof is rather standard and it is based on the injectivity of ι, formulas (2.20), (2.23) and Theorem 1.1. □

Our next goal is to apply Theorem 1.4 for studying the completeness of root functions of weak perturbations of Q_j. Lemma 2.4 and Theorem 2.5 show sufficient conditions for the inclusion (2.9) to be compact. However, we should also describe typical situations where the operators Q_1, Q_2, Q_3 have finite order. With this purpose, we present a broad class of finite order compact operators acting in spaces of integrable functions. The following result goes back at least as far as [Agm62].

Theorem 3.2. *Let $s \in \mathbb{R}$ and $A : H^s(\mathcal{D}) \to H^s(\mathcal{D})$ be a compact operator. If there is $\delta s > 0$ such that A maps $H^s(\mathcal{D})$ continuously to $H^{s+\delta s}(\mathcal{D})$, then it belongs to Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for each $\varepsilon > 0$.*

Proof. For the case $s \in \mathbb{Z}_{\geq 0}$ see [Agm62]. For the case $s \in \mathbb{R}$ and Sobolev spaces on a compact closed manifold $\mathcal{D}$ see Proposition 5.4.1 in [Agr90]. To the best of our knowledge, no proof has been appeared for the general case. So we indicate crucial steps of the proof.

Let Q be the cube
$$Q = \{x \in \mathbb{R}^n : |x_j| < \pi, \ j = 1, \ldots, n\}$$
in $\mathbb{R}^n$. Given a function $u \in L^2(Q)$, we consider the Fourier series expansion
$$u(x) \sim \sum_{k \in \mathbb{Z}^n} c_k(u)\, e^{\iota \langle k,x \rangle}$$
and introduce the norm
$$\|u\|^2_{H^{(s)}} = |a_0(u)|^2 + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{2s}\, |c_k(u)|^2,$$
where s is a non-negative real number. The subspace of functions for which this norm is finite is denoted by $H^{(s)}$. Obviously, $H^{(s)}$ is a Hilbert space which, for non-negative integral s, can be regarded as a closed subspace of the Sobolev space $H^s(Q)$. We see readily that $H^s_{\mathrm{comp}}(Q) \mapsto H^{(s)}$. For $s < 0$, we write $H^{(s)}$ for the dual of $H^{(-s)}$ with respect to the sesquilinear pairing $(\cdot, \cdot)$ induced by the inner product $(\cdot, \cdot)_{H^{(0)}}$. The norm in $H^{(s)}$ is still given by the same formula, as is easy to check.

Without loss of the generality we can assume that the closure of $\mathcal{D}$ is situated in the cube Q. For $s \geq 0$, we denote by $r_{s,\mathcal{D}}$ the restriction operator from $H^s(Q)$ to $H^s(\mathcal{D})$. By the above, $r_{s,\mathcal{D}}$ acts to the elements of $H^{(s)}$, too, mapping these continuously to $H^s(\mathcal{D})$. As the boundary of $\mathcal{D}$ is Lipschitz, for each $s \in \mathbb{Z}_{\geq 0}$ there is a bounded extension operator $e_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^s_{\mathrm{comp}}(Q)$ (see for instance [Bur98, Ch. 6]). We will think of $e_{s,\mathcal{D}}$ as bounded linear operator from $H^s(\mathcal{D})$ to $H^{(s)}$, provided that $s \in \mathbb{Z}_{\geq 0}$.

Given any integer $s \geq 0$, an interpolation procedure applies to the pair $(H^s(\mathcal{D}), H^{s+1}(\mathcal{D}))$, thus giving a family of function spaces in $\mathcal{D}$ of fractional smoothness $(1 - \vartheta)s + \vartheta(s + 1) = s + \vartheta$ with $0 < \vartheta < 1$. The Banach spaces obtained in this way coincide with $H^{s+\vartheta}(\mathcal{D})$ up to equivalent norms. Thus, we can apply interpolation arguments to conclude that there is a bounded linear extension operator $e_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^{(s)}$ for all real $s \geq 0$. By construction,

$$r_{s,\mathcal{D}}\, e_{s,\mathcal{D}} u = u \tag{3.1}$$

holds for each $u \in H^s(\mathcal{D})$ with $s \geq 0$.

For $s < 0$ we introduce the mappings

$$\begin{array}{rccc} r_{s,\mathcal{D}} & : & H^{(s)} & \to & H^s(\mathcal{D}), \\ e_{s,\mathcal{D}} & : & H^s(\mathcal{D}) & \to & H^{(s)}, \end{array}$$

using the duality between the spaces $H^{(s)}$ and $H^{(-s)}$. Namely, if $s < 0$ we set

$$\begin{array}{rcl} (r_{s,\mathcal{D}} u, v) & := & (u, e_{-s,\mathcal{D}} v), \\ (e_{s,\mathcal{D}} u, v) & := & (u, r_{-s,\mathcal{D}} v) \end{array} \tag{3.2}$$

for all $u \in H^{(s)}$, $v \in H^{-s}(\mathcal{D})$ and for all $u \in H^s(\mathcal{D})$, $v \in H^{(-s)}$, respectively. As

$$|(u, e_{-s,\mathcal{D}} v)| \leq \|u\|_{H^{(s)}}\, \|e_{-s,\mathcal{D}}\|\, \|v\|_{H^{-s}(\mathcal{D})}$$

for all $u \in H^{(s)}$ and $v \in H^{-s}(\mathcal{D})$, which is a consequence of duality between the spaces $H^{(s)}$ and $H^{(-s)}$, the first identity of (3.2) defines a bounded linear operator $r_{s,\mathcal{D}} : H^{(s)} \to H^s(\mathcal{D})$ indeed. Similarly, by the duality between $H^s(\mathcal{D})$ and $H^{-s}(\mathcal{D})$ (cf. Lemma 2.3), the second identity of (3.2) defines a bounded linear operator $e_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^{(s)}$.

On applying equality (3.1) we get $(r_{s,\mathcal{D}}\, e_{s,\mathcal{D}} u, v) = (u, v)$ for all $u \in H^s(\mathcal{D})$ and $v \in H^{-s}(\mathcal{D})$ with real $s < 0$. In other words, the operators $r_{s,\mathcal{D}}$ and $r_{s,\mathcal{D}}$ satisfy (3.1) for all $s \in \mathbb{R}$, i.e.,

$$r_{s,\mathcal{D}}\, e_{s,\mathcal{D}} = I_{H^s(\mathcal{D})}. \tag{3.3}$$

For $t > s$ we denote by

$$\begin{array}{rccc} \iota_{t,s,\mathcal{D}} & : & H^t(\mathcal{D}) & \to & H^s(\mathcal{D}), \\ \iota_{t,s} & : & H^{(t)} & \to & H^{(s)} \end{array}$$

the natural inclusion mappings. If $t < 0$, by this is meant

$$\begin{array}{rcl} (\iota_{t,s,\mathcal{D}} u, v) & = & (u, \iota_{-s,-t,\mathcal{D}} v), \\ (\iota_{t,s} u, v) & = & (u, \iota_{-s,-t,\mathcal{D}} v) \end{array} \tag{3.4}$$

for all $u \in H^t(\mathcal{D})$, $v \in H^{-s}(\mathcal{D})$ and $u \in H^{(t)}$, $v \in H^{(-s)}$, respectively. It is clear that

$$\begin{array}{rcl} r_{s,\mathcal{D}}\, \iota_{t,s} & = & \iota_{t,s,\mathcal{D}}\, r_{t,\mathcal{D}}, \\ \iota_{t,s}\, e_{t,\mathcal{D}} & = & e_{s,\mathcal{D}}\, \iota_{t,s,\mathcal{D}}, \\ r_{s,\mathcal{D}}\, \iota_{t,s}\, e_{t,\mathcal{D}} & = & \iota_{t,s,\mathcal{D}} \end{array} \tag{3.5}$$

provided $t \geq 0$. If $t < 0$ then combining (3.2), (3.4) and (3.5) yields

$$\begin{array}{rcl} (r_{s,\mathcal{D}}\, \iota_{t,s} u, v) & = & (u, \iota_{-s,-t}\, e_{-s,\mathcal{D}} v) \\ & = & (u, e_{-t,\mathcal{D}}\, \iota_{-s,-t,\mathcal{D}} v\rangle \\ & = & (\iota_{t,s,\mathcal{D}}\, r_{t,\mathcal{D}} u, v) \end{array}$$

for all $u \in H^{(t)}$ and $v \in H^{-s}(\mathcal{D})$, and

$$
\begin{array}{rcl}
(\iota_{t,s}\, e_{t,\mathcal{D}} u, v) & = & (u, r_{-t,\mathcal{D}}\, \iota_{-s,-t} v) \\
& = & (u, \iota_{-s,-t}\, r_{-s,\mathcal{D}} v) \\
& = & (e_{s,\mathcal{D}}\, \iota_{t,s,\mathcal{D}} u, v)
\end{array}
$$

for all $u \in H^t(\mathcal{D})$ and $v \in H^{(-s)}$, whence $r_{s,\mathcal{D}}\, \iota_{t,s}\, e_{t,\mathcal{D}} = \iota_{t,s,\mathcal{D}}$. Therefore, equalities (3.5) are valid not only for real $t \geq 0$ but also for all $t \in \mathbb{R}$.

Lemma 3.3. *Let $s \in \mathbb{R}$ and $K : H^{(s)} \to H^{(s)}$ be a compact operator. If there is $\delta s > 0$ such that K maps $H^{(s)}$ continuously to $H^{(s+\delta s)}$, then K is of Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for each $\varepsilon > 0$.*

Proof. Put

$$
\Lambda_r u\,(x) = \sum_{k\in\mathbb{Z}^n} (1+|k|^2)^{r/2}\, c_k(u)\, e^{\imath\langle k,x\rangle}.
$$

Obviously, Λ_r maps $H^{(s)}$ continuously to $H^{(s-r)}$ for all $s \in \mathbb{R}$. Für each fixed s, the operator $\Lambda_{-r}\, \iota_{s+r,s}$ is selfadjoint and compact in $H^{(s+r)}$. Its eigenvalues are $(1+|k|^2)^{-r/2}$ and the corresponding eigenfunctions are $e^{\imath\langle k,x\rangle}$. The series

$$
\sum_{k\in\mathbb{Z}^n} (1+|k|^2)^{-pr/2}
$$

(counting the eigenvalues with their multiplicities) converges for all $p > n/r$, and so $\Lambda_{-r}\, \iota_{s+r,s}$ is of Schatten class $\mathfrak{S}_{n/r+\varepsilon}$ for any $\varepsilon > 0$.

Obviously, $\Lambda_{-r}\Lambda_r = I$ holds for all $r > 0$. By assumption, the operator K factors through the embedding $\iota_{s+\delta s,s} : H^{(s+\delta s)} \to H^{(s)}$, i.e., there is a bounded linear operator $K_0 : H^{(s)} \to H^{(s+\delta s)}$ such that $K = \iota_{s+\delta s,s} K_0$. Then

$$
\begin{array}{rcl}
K & = & \Lambda_{-\delta s}\, \Lambda_{\delta s} K \\
& = & \Lambda_{-\delta s}\, \Lambda_{\delta s}\, \iota_{s+\delta s,s} K_0 \\
& = & \Lambda_{-\delta s}\, \iota_{s+\delta s,s}\, \Lambda_{\delta s} K_0.
\end{array}
$$

Since the operator $\Lambda_{\delta s}\, K_0 : H^{(s)} \to H^{(s)}$ is bounded, Lemma 1.3 implies that K belongs to the Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for any $\varepsilon > 0$. $\square$

We are now in a position to complete the proof of Theorem 3.2. Suppose that $A_0 : H^s(\mathcal{D}) \to H^{s+\delta s}(\mathcal{D})$ is a bounded linear operator, such that $A = \iota_{s+\delta s,s} A_0$. Set

$$
K_0 = e_{s+\delta s,\mathcal{D}}\, A_0\, r_{s,\mathcal{D}},
$$

then K_0 maps $H^{(s)}$ continuously to $H^{(s+\delta s)}$. By Lemma 3.3, the composition $K = \iota_{s+\delta s,s} K_0$ is of Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for any $\varepsilon > 0$. Besides, we get

$$
r_{s+\delta s,\mathcal{D}}\, K_0 = A_0\, r_{s,\mathcal{D}} \tag{3.6}
$$

because of (3.3).

Let λ be a non-zero eigenvalue of A and $u \in H^s(\mathcal{D})$ a root function corresponding to λ, i.e., $(A-\lambda I)^m u = 0$ for some natural number m. Then, using (3.5) and (3.6), we conclude that

$$
\begin{array}{rcl}
(K-\lambda I)^m\, e_{s,\mathcal{D}} u & = & e_{s,\mathcal{D}}\, (A-\lambda I)^m u \\
& = & 0,
\end{array}
$$

that is each non-zero eigenvalue of A is actually an eigenvalue for K of the same multiplicity. Therefore, A belongs to the Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for any $\varepsilon > 0$, too. $\square$

Corollary 3.4. *If for some $s > 0$ there is a continuous embedding*

$$\iota_s : H_{SL}(\mathcal{D}) \hookrightarrow H^s(\mathcal{D}) \tag{3.7}$$

then any compact operator $R : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ which maps $H^-_{SL}(\mathcal{D})$ continuously to $H_{SL}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$. In particular, its order is finite.

Proof. It follows from Theorem 3.2 in a standard way (cf. [Agm62]). Namely, let R_0 be the operator R which is thought of as a bounded map of $H^-_{SL}(\mathcal{D})$ to $H_{SL}(\mathcal{D})$. If $i : H^s(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ is the natural inclusion and $i' : L^2(\mathcal{D}) \hookrightarrow H^{-s}(\mathcal{D})$ is the transposed map then it is easily seen that the operator $i' i \,\iota_s R_0 \iota'_s : H^{-s}(\mathcal{D}) \to H^{-s}(\mathcal{D})$ belongs to the Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$ and it has the same eigenvectors and eigenvalues (with the same multiplicities!) as the operator R. $\square$

Corollary 3.5. *Suppose there is a continuous embedding (3.7) with some $s > 0$. Then the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$ (and so they are of finite order).*

Lemma 2.4 and Theorem 2.5 provide sufficient conditions for a continuous embedding (3.7) to be true with $s = 1$ and $0 < s < 1/2$, respectively.

Theorem 3.6. *If the operator $Q_1 : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is of finite order then, for any invertible operator of the type $L_0 + \delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ with a compact operator $\delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$, the system of root functions of the compact operator*

$$P_1 = \iota' \iota \,(L_0 + \delta L)^{-1} : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$$

is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.

Proof. The proof is based on Theorem 1.4 and the equality

$$L_0^{-1} - (L_0 + \delta L)^{-1} \quad = \quad L_0^{-1} \left(\delta L \,(L_0 + \delta L)^{-1} \right), \tag{3.8}$$

where the operator

$$\delta L \,(L_0 + \delta L)^{-1} : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$$

is compact (cf. Proposition 6.1 of [Agr11a] and [Agr11c, p. 12]). $\square$

Similar assertions are also true for the weak perturbations of the operators Q_2 and Q_3.

The operator $L_0 + \delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ with a compact operator δL fails to be injective in general, and so Theorem 3.6 does not apply. However, as L_0 is continuously invertible, we conclude that $L = L_0 + \delta L$ is Fredholm. In particular, there is a constant c, such that

$$\|u\|_{SL} \leq c \left(\|Lu\|_{H^-_{SL}(\mathcal{D})} + \|u\|_{H^-_{SL}(\mathcal{D})} \right) \tag{3.9}$$

for all $u \in H_{SL}(\mathcal{D})$.

We next extend Theorem 3.6 to Fredholm operators. To this end denote by T the unbounded linear operator $H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ with domain $\mathcal{D}_T = H_{SL}(\mathcal{D})$ which maps an element $u \in \mathcal{D}_T$

to Lu. The operator T is clearly closed because of inequality (3.9). It is densely defined as $H^1(\mathcal{D},S) \subset H_{SL}(\mathcal{D})$ is dense in $H^-_{SL}(\mathcal{D})$. It is well known that the null space of T is finite dimensional in $H_{SL}(\mathcal{D})$ and its range is closed in $H^-_{SL}(\mathcal{D})$.

When speaking on eigen- and root functions u of the operator T we always assume that $u \in \mathcal{D}_T$ and $(T-\lambda I)^j u \in \mathcal{D}_T$ for all $j = 1, \ldots, m-1$.

Let $T_0 : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ correspond to the selfadjoint operator L_0. The operator T_0 is obviously continuously invertible and the inverse operator coincides with $\iota'\iota\, L_0^{-1} = Q_1$.

Lemma 3.7. *The spectrum of the operator T_0 consists of the points $\mu_\nu = \lambda_\nu^{-1}$ in $\mathbb{R}_{>0}$, where λ_ν are the eigenvalues of Q_1.*

If the spectrum of the operator T fails to be the whole complex plane, i.e., if the resolvent $\mathcal{R}(\lambda;T) = (T-\lambda I)^{-1}$ exists for some $\lambda = \lambda_0$, then it follows from the resolvent equation (since $\mathcal{R}(\lambda_0;T)$ is compact) that $\mathcal{R}(\lambda;T)$ exists for all $\lambda \in \mathbb{C}$ except for a discrete sequence of points $\{\lambda_\nu\}$ which are the eigenvalues of T (see [Kel71, p. 17]. In the general case, however, one cannot exclude the situation where the spectrum of T is the whole complex plane.

Theorem 3.8. *Suppose that $\delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is a compact operator and that the operator $Q_1 : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is of finite order. Then the spectrum of the closed operator $T : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ corresponding to $L = L_0 + \delta L$, is different from $\mathbb{C}$ and the system of root functions of T is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. Moreover, for any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg\lambda| < \varepsilon$.*

Proof. First we note that

$$T - \lambda I = L - \lambda\, \iota'\iota \tag{3.10}$$

on $H_{SL}(\mathcal{D})$ for all $\lambda \in \mathbb{C}$. Let us prove that there is $N \in \mathbb{N}$, such that $\lambda_0 = -N$ is a resolvent point of T. For this purpose, using (3.10) and Lemma 3.7, we get

$$T + kI = (I + \delta L(L_0 + k\,\iota'\iota)^{-1})(T_0 + kI) \tag{3.11}$$

for all $k \in \mathbb{N}$.

We will show that the operator $I + \delta L(L_0 + k\,\iota'\iota)^{-1}$ is injective for some $k \in \mathbb{N}$. Indeed, we argue by contradiction. Suppose for any $k \in \mathbb{N}$ there is $f_k \in H^-_{SL}(\mathcal{D})$, such that $\|f_k\|_{H^-_{SL}(\mathcal{D})} = 1$ and

$$(I + \delta L(L_0 + k\,\iota'\iota)^{-1}) f_k = 0. \tag{3.12}$$

Given any $u \in H_{SL}(\mathcal{D})$ and $k \in \mathbb{N}$, an easy computation shows that

$$\begin{aligned} \|(L_0 + k\,\iota'\iota)u\|^2_{H^-_{SL}(\mathcal{D})} &= \|u + k\, L_0^{-1} u\|^2_{SL} \\ &= \|u\|^2_{SL} + 2k\,\|u\|^2_{L^2(\mathcal{D})} + k^2\,\|L_0^{-1}u\|^2_{SL} \\ &\geq \|u\|^2_{SL}. \end{aligned}$$

Hence, the sequence $u_k := (L_0 + k\,\iota'\iota)^{-1} f_k$ is bounded in $H_{SL}(\mathcal{D})$. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{f_{k_j}\}$ with the property that both $\{f_{k_j}\}$ and $\{u_{k_j}\}$ converge weakly in the spaces $H^-_{SL}(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to limits f and u, respectively. Since δL is compact, it follows that the sequence $\{\delta L u_{k_j}\}$ converges to $\delta L u$ in $H^-_{SL}(\mathcal{D})$, and so $\{f_{k_j}\}$ converges to f because of (3.12). Obviously,

$$\|f\|_{H^-_{SL}(\mathcal{D})} = 1.$$

In particular, we conclude that the sequence $\{\delta L(L_0 + k_j\,\iota'\iota)^{-1} f_{k_j}\}$ converges to $-f$ whence

$$f = -\delta L\, u. \tag{3.13}$$

Further, on passing to the weak limit in the equality $f_{k_j} = (L_0 + k_j\,\iota' \circ \iota)u_{k_j}$ we obtain

$$f = L_0 u - \lim_{k_j \to \infty} k_j\,\iota'\iota\, u_{k_j},$$

for the continuous operator $L_0 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ maps weakly convergent sequences to weakly convergent sequences. As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\, u_{k_j}\}$ converges to $\iota'\iota\, u$ in the space $H^-_{SL}(\mathcal{D})$ and $\iota'\iota\, u \neq 0$ which is a consequence of (3.13) and the injectivity of $\iota'\iota$. This shows readily that the weak limit

$$\lim_{k_j \to \infty} k_j\,\iota'\iota\, u_{k_j} = L_0 u - f$$

does not exist, a contradiction.

We have proved more, namely that the operator $I + \delta L(L_0 + k\,\iota'\iota)^{-1}$ is injective for all but a finitely many natural numbers k. Since this is a Fredholm operator of index zero, it is continuously invertible. Hence, (3.11) and Lemma 3.7 imply that $(T - \lambda_0 I)^{-1}$ exists for some $\lambda_0 = -N$ with $N \in \mathbb{N}$.

As λ_0 is a resolvent point of T,

$$(T - \lambda_0 I)^{-1} = (L - \lambda_0\,\iota'\iota)^{-1}$$

on $H^-_{SL}(\mathcal{D})$. Since $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is Fredholm and the inclusion ι compact, the operator $L - \lambda_0\,\iota'\iota : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is Fredholm. So $(L - \lambda_0\,\iota'\iota)^{-1}$ maps $H^-_{SL}(\mathcal{D})$ continuously to $H_{SL}(\mathcal{D})$. Similarly to (3.8) we get

$$L_0^{-1} - (L - \lambda_0\,\iota'\iota)^{-1} = L_0^{-1}\left((\delta L - \lambda_0\,\iota'\iota)(L - \lambda_0\,\iota'\iota)^{-1}\right).$$

Then, Theorem 3.6 yields that the root functions $\{u_\nu\}$ of the operator $(L - \lambda_0\,\iota'\iota)^{-1}$ are complete in the spaces $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$.

From (3.10) it follows that the systems of root functions related to the operators $(L - \lambda_0\,\iota'\iota)^{-1}$ and $T - \lambda_0\, I$ coincide.

Finally, as the operators $T - \lambda_0 I$ and T have the same root functions, we conclude that $\mathcal{L}(\{u_\nu\})$ is dense in the spaces $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$. □

The equality $(T - \lambda I)u = 0$ for a function $u \in H_{SL}(\mathcal{D})$ may be equivalently reformulated by saying that u is a solution in a weak sense to the boundary value problem

$$\left\{\begin{array}{rclcl} Au & = & \lambda u & \text{in} & \mathcal{D}, \\ Bu & = & 0 & \text{on} & \partial\mathcal{D}, \end{array}\right. \tag{3.14}$$

where the pair (A, B) corresponds to the perturbation $L_0 + \delta L$. For $n = 1$ such problems are known as Sturm-Liouville boundary problems for second order ordinary differential equations (see for instance [Har64, Ch. XI, § 4]). Thus, we may still refer to (3.14) as the Sturm-Liouville problem in many dimensions.

Now we want to study the completeness of root functions of "small" perturbations of compact selfadjoint operators instead of the weak ones. To this end we apply the so-called method of rays of minimal growth of resolvent which leads to more general results than Theorem 1.4. This idea seems to go back at least as far as [Agm62].

4. Rays of minimal growth

We first describe briefly the method of minimal growth rays following [DS63] and Theorem 6.1 of [GK69, p. 302].

Let $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ be the bounded linear operator constructed in Section 2. We still assume that estimates (2.7) and (2.19) hold and that the operator L is Fredholm. In the sequel we confine ourselves to those Sturm-Liouville problems for which the spectrum of the corresponding unbounded closed operator $T : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is discrete, cf. [Agm62]. We denote by $\mathcal{R}(\lambda; T)$ the resolvent of the operator T.

Definition 4.1. A ray $\arg\lambda = \vartheta$ in the complex plane $\mathbb{C}$ is called a ray of minimal growth of the resolvent $\mathcal{R}(\lambda; T) : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ if the resolvent exists for all λ of sufficiently large modulus on this ray, and if, moreover, for all such λ an estimate

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \le c\, |\lambda|^{-1} \tag{4.1}$$

holds with a constant $C > 0$.

Theorem 4.2. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$. Suppose there are rays of minimal growth of the resolvent* $\arg\lambda = \vartheta_j$*, where $j = 1, \ldots, J$, in the complex plane, such that the angles between any two neighbouring rays are less than $2\pi s/n$. Then the spectrum of the operator T is discrete and the root functions form a complete system in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.*

Proof. The proof actually follows by the same method as that in Theorem 3.2 of [Agm62], see also Theorem 6.1 in [GK69, p. 302]. □

This theorem raises the question under what conditions neighbouring rays of minimal growth are close enough. We now indicate some conditions for a ray $\arg\lambda = \vartheta$ in the complex plane to be a ray of minimal growth for the resolvent of T.

Lemma 4.3. *Each ray* $\arg\lambda = \vartheta$ *with $\vartheta \neq 0$ is a ray of minimal growth for $\mathcal{R}(\lambda; T_0)$ and*

$$\|(T_0 - \lambda I)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \le \begin{cases} (|\lambda|\,|\sin(\arg\lambda)|)^{-1}, & \textit{if} \quad |\arg\lambda| \in (0, \pi/2), \\ |\lambda|^{-1}, & \textit{if} \quad |\arg\lambda| \in [\pi/2, \pi]. \end{cases} \tag{4.2}$$

Moreover, the operator $L_0 - \lambda\,\iota'\iota : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is continuously invertible and

$$\|(L_0 - \lambda\,\iota'\iota)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}), H_{SL}(\mathcal{D}))} \le \begin{cases} |\sin(\arg\lambda)|, & \textit{if} \quad |\arg\lambda| \in (0, \pi/2), \\ 1, & \textit{if} \quad |\arg\lambda| \in [\pi/2, \pi]. \end{cases} \tag{4.3}$$

Proof. According to Lemma 3.7 the resolvent

$$(T_0 - \lambda I)^{-1} : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$$

exists for all $\lambda \in \mathbb{C}$ away from the positive real axis. As the operator $Q_3 = L_0^{-1}\,\iota'\iota$ is selfadjoint, the operator T_0 is symmetric, i.e.,

$$\begin{aligned} (T_0 u, g)_{H^-_{SL}(\mathcal{D})} &= (L_0 u, g)_{H^-_{SL}(\mathcal{D})} \\ &= (u, Q_3^{-1} g)_{SL} \\ &= (Q_3^{-1} u, g)_{SL} \\ &= (u, L_0 g)_{H^-_{SL}(\mathcal{D})} \\ &= (u, T_0 g)_{H^-_{SL}(\mathcal{D})} \end{aligned}$$

for all $u, v \in H_{SL}(\mathcal{D})$. If $|\arg(\lambda)| \in (0, \pi/2)$, then

$$\begin{aligned} \|(T_0 - \lambda I)u\|^2_{H^-_{SL}(\mathcal{D})} &= \|(T_0 - \Re\lambda\, I)u\|^2_{H^-_{SL}(\mathcal{D})} + |\Im\lambda|^2\, \|u\|^2_{H^-_{SL}(\mathcal{D})} \\ &\geq |\lambda|^2 |\sin(\arg\lambda)|^2\, \|u\|^2_{H^-_{SL}(\mathcal{D})} \end{aligned}$$

for all $u \in H_{SL}(\mathcal{D})$, which establishes the first estimate of (4.2). If $|\arg\lambda| \in [\pi/2, \pi]$, then $\Re\lambda \leq 0$ whence

$$\|(T_0 - \lambda I)u\|^2_{H^-_{SL}(\mathcal{D})} \geq |\lambda|^2\, \|u\|^2_{H^-_{SL}(\mathcal{D})}$$

and so the second estimate of (4.2) holds.

Now it follows from (3.10) that the operator $L_0 - \lambda\, \iota'\iota$ is injective for $\lambda \in \mathbb{C}$ away from the positive real axis. As this operator is Fredholm and its index is zero, it is continuously invertible. Finally, as the operator $Q_3 = L_0^{-1}\, \iota'\iota$ is positive, we deduce readily that

$$\begin{aligned} \|(L_0 - \lambda\, \iota'\iota)u\|_{H^-_{SL}(\mathcal{D})} &= \|(I - \lambda L_0^{-1}\, \iota'\iota)u\|_{SL} \\ &\geq |\lambda|\, |\Im\lambda^{-1}|\, \|u\|_{SL} \\ &= |\sin(\arg\lambda)|\, \|u\|_{SL}, \end{aligned}$$

if $|\arg\lambda| \in (0, \pi/2)$, i.e., the second estimate of (4.3) is fulfilled. Similar arguments lead to the second estimate of (4.3). □

Theorem 4.4. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$ and estimate (2.19) be fulfilled with a constant $c < |\sin(\pi s/n)|$. Then all eigenvalues of the closed operator $T : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ belong to the corner $|\arg\lambda| \leq \arcsin c$, each ray $\arg\lambda = \vartheta$ with $|\vartheta| > \arcsin c$ is a ray of minimal growth for $\mathcal{R}(\lambda; T)$ and the system of root functions is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.*

Proof. First we note that, by Lemma 2.7, the operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is invertible. Indeed, $L = L_0 + \delta L$ where $\delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is a bounded operator with the norm $\|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H^-_{SL}(\mathcal{D}))} < 1 = \|L_0^{-1}\|^{-1}$. In particular, by (3.10), the spectrum of the corresponding operator T does not coincide with the whole complex plane.

Fix $\vartheta \neq 0$ and set $m_\vartheta = |\sin\vartheta|$, if $|\vartheta| \in (0, \pi/2)$, and $m_\vartheta = 1$, if $|\vartheta| \in [\pi/2, \pi]$. If $m_\vartheta > c$ then

$$\|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H^-_{SL}(\mathcal{D}))} \leq c < m_\vartheta \leq \|(L_0 - \lambda\, \iota'\iota)^{-1}\|^{-1}_{\mathcal{L}(H_{SL}(\mathcal{D}), H^-_{SL}(\mathcal{D}))}.$$

Hence it follows that the operator $L - \lambda\, \iota'\iota : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is continuously invertible and

$$\|(L - \lambda\, \iota'\iota)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}), H_{SL}(\mathcal{D}))} \leq (m_\vartheta - c)^{-1}. \tag{4.4}$$

In order to establish estimate (4.1) we have to show that there is a constant $C > 0$, such that

$$C\, |\lambda|^{-1}\, \|(T - \lambda I)u\|_{H^-_{SL}(\mathcal{D})} \geq \|u\|_{H^-_{SL}(\mathcal{D})}$$

for all $u \in H_{SL}(\mathcal{D})$.

If $\arg\lambda = \vartheta$ with $m_\vartheta > c$, then, by (3.10), we get

$$\begin{aligned} \|(T - \lambda I)u\|_{H^-_{SL}(\mathcal{D})} &= \|(L - \lambda\, \iota'\iota)u\|_{H^-_{SL}(\mathcal{D})} \\ &\geq (m_\vartheta - c)\, \|u\|_{SL} \\ &\geq (m_\vartheta) - c)\, \|u\|_{H^-_{SL}(\mathcal{D})} \end{aligned}$$

for all $u \in H_{SL}(\mathcal{D})$. Therefore, given any λ on the ray $\arg\lambda = \vartheta$ with $m_\vartheta > c$, it follows that

1) The range of the operator $T - \lambda I : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is a closed subspace of $H^-_{SL}(\mathcal{D})$.

2) The null space of the operator $T - \lambda I : H^-_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is trivial.

By (3.10) the range of $T - \lambda I$ coincides with the range of $L - \lambda\, \iota' \iota$ which is the whole space $H^-_{SL}(\mathcal{D})$. Hence, the resolvent $(T - \lambda I)^{-1}$ exists for all λ away from the corner $|\arg\lambda| \leq \arcsin c$ in the complex plane. On applying (3.10) and Lemma 4.3 we obtain

$$T - \lambda I = L_0 + \delta L - \lambda\, \iota' \iota = (I + \delta L (L_0 - \lambda\, \iota' \iota)^{-1})(T_0 - \lambda I) \tag{4.5}$$

on $H_{SL}(\mathcal{D})$ and

$$\begin{array}{cl} & \|(I + \delta L (L_0 - \lambda\, \iota' \iota)^{-1}) u\|_{H^-_{SL}(\mathcal{D})} \\ \geq & \|u\|_{H^-_{SL}(\mathcal{D})} - \|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}),(H^-_{SL}(\mathcal{D})))} \|(L_0 - \lambda\, \iota' \iota)^{-1} u\|_{H_{SL}(\mathcal{D})} \\ \geq & (1 - c/m_\vartheta)\, \|u\|_{H^-_{SL}(\mathcal{D})}. \end{array}$$

Therefore the operator $I + \delta L (L_0 - \lambda\, \iota' \iota)^{-1}$ is continuously invertible as Fredholm operator of zero index and trivial null space. Moreover,

$$\|(I + \delta L (L_0 - \lambda\, \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \leq (1 - c/m_\vartheta)^{-1}.$$

Now (4.5) implies

$$\begin{array}{cl} & \|(T - \lambda I)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \\ \leq & \|(I + \delta L (L_0 - \lambda\, \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \|(T_0 - \lambda\, I)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \\ \leq & (1 - c/m_\vartheta)^{-1} m_\vartheta^{-1} |\lambda|^{-1} \end{array} \tag{4.6}$$

for all λ satisfying $\arg\lambda = \vartheta$ with $m_\vartheta > c$.

Thus, all rays outside of the corner $|\arg\lambda| \leq \arcsin c$ are rays of minimal growth. By the hypothesis of the theorem, the angles between the pairs of neighbouring rays $\arg\lambda = \vartheta$, are less than $2\pi s/n$, and so the completeness of root functions follows from Theorem 4.2. □

We are now in a position to prove the main result of this section. When compared with [Agr11c] our contribution consists in developing dual function spaces which fit the problem including the non-coercive case.

Theorem 4.5. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$, $\delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ be a bounded linear operator whose norm is less then $|\sin(\pi s/n)$, and $C : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ be compact. Then the following is true:*

1) The spectrum of the operator T in $H^-_{SL}(\mathcal{D})$ corresponding to $L_0 + \delta L + C$ is discrete.

2) For any $\varepsilon > 0$, all eigenvalues of the operator T (except for a finite number) belong to the corner $|\arg\lambda)| < \arcsin \|\delta L\| + \varepsilon$.

3) Each ray $\arg\lambda = \vartheta$ with

$$|\vartheta| > \arcsin \|\delta L\| \tag{4.7}$$

is a ray of minimal growth for $\mathcal{R}(\lambda; T)$.

4) The system of root functions is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.

Proof. First we note that the operator $L_0 + \delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is continuously invertible and hence the operator $L_0 + \delta L + C : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ is actually Fredholm.

Theorem 4.4 implies that all rays satisfying (4.7) are rays of minimal growth for $\mathcal{R}(\lambda; T_0 + \delta T)$ with the closed operator $T_0 + \delta T$ in $H^-_{SL}(\mathcal{D})$ corresponding to $L_0 + \delta L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$.

Fix an arbitrary $\varepsilon > 0$. Then estimates (4.4) and (4.6) imply that there are constants c_1 and c_2 depending on ε, such that

$$\begin{aligned} \|(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}), H_{SL}(\mathcal{D}))} &\leq c_1, & (4.8)\\ \|(T_0 + \delta T - \lambda I)^{-1}\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} &\leq c_2\, |\lambda|^{-1} & (4.9)\end{aligned}$$

for all λ satisfying

$$|\arg \lambda| \geq \arcsin \|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H^-_{SL}(\mathcal{D}))} + \varepsilon. \tag{4.10}$$

Then, using (3.10), (4.8) and Theorem 4.4 we obtain

$$T - \lambda I = \Big(I + C(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}\Big)(T_0 + \delta T - \lambda I) \tag{4.11}$$

on $H_{SL}(\mathcal{D})$ for all rays satisfying (4.10).

We now prove that there is a constant $M_\varepsilon > 0$ depending on ε, such that the operator $I + C(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}$ is injective for all λ satisfying both (4.10) and $|\lambda| \geq M_\varepsilon$. To do this, we argue by contradiction in the same way as in the proof of Theorem 3.8. Suppose for each natural number k there are $f_k \in H^-_{SL}(\mathcal{D})$, satisfying $\|f_k\|_{H^-_{SL}(\mathcal{D})} = 1$, and λ_k, satisfying (4.10) and $|\lambda_k| \geq k$, such that

$$\Big(I + C(L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1}\Big) f_k = 0. \tag{4.12}$$

It follows from (4.8) that the sequence $u_k = (L_0 + \delta L - \lambda_k \iota'\iota)^{-1} f_k$ is bounded in $H_{SL}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces one can assume without restriction of generality that the sequences $\{f_k\}$ and $\{u_k\}$ converge weakly in the spaces $H^-_{SL}(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to functions f and u, respectively. Since C is compact, it follows that the sequence $\{Cu_k\}$ converges to Cu in $H^-_{SL}(\mathcal{D})$ and so $\{f_k\}$ converges to f, which is due to (4.12). Obviously, the $H^-_{SL}(\mathcal{D})$-norm of f just amounts to 1. In particular, we conclude that $\{C(L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1}) f_k\}$ converges to $-f$ whence

$$f = -Cu. \tag{4.13}$$

Further, as $f_k = (L_0 + \delta L - \lambda_k\, \iota'\iota)\, u_k$, letting $k \to \infty$ in this formula yields readily

$$f = (L_0 + \delta L)u - \lim_{k \to \infty} \lambda_k\, \iota'\iota\, u_k.$$

As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\, u_k\}$ converges to $\iota'\iota\, u$ in the space $H^-_{SL}(\mathcal{D})$, and $\iota'\iota u \neq 0$ because of (4.13) and the injectivity of $\iota'\iota$. Therefore, the weak limit

$$\lim_{k \to \infty} \lambda_k\, \iota'\iota\, u_k = (L_0 + \delta L)u - f$$

fails to exist, for $\{\lambda_k\}$ is unbounded. A contradiction.

As the operator $I + C(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}$ is Fredholm and it has index zero, this operator is continuously invertible for all $\lambda \in \mathbb{C}$ satisfying both (4.10) and $|\lambda| \geq M_\varepsilon$. Set

$$N_\varepsilon = \inf \|(I + C(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}) f\|_{H^-_{SL}(\mathcal{D})} \geq 0,$$

the infimum being over all $f \in H^-_{SL}(\mathcal{D})$ of norm 1 and all $\lambda \in \mathbb{C}$ satisfying (4.10) and $|\lambda| \geq M_\varepsilon$. We claim that $N_\varepsilon > 0$. To show this, we argue by contradiction. If $N_\varepsilon = 0$ then there are sequences $\{f_k\}$ in $H^-_{SL}(\mathcal{D})$, each f_k being of norm 1, and $\{\lambda_k\}$ satisfying (4.10) and $|\lambda| \geq M_\varepsilon$, such that

$$\lim_{k\to\infty} \|(I + C(L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1})f_k\|_{H^-_{SL}(\mathcal{D})} = 0. \tag{4.14}$$

Again, by (4.8), the sequence $u_k = (L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1} f_k$ is bounded in $H_{SL}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces we may assume that the sequences $\{f_k\}$ and $\{u_k\}$ are weakly convergent in the spaces $H^-_{SL}(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to functions f and u, respectively. Since C is compact, the sequence $\{Cu_k\}$ converges to Cu in $H^-_{SL}(\mathcal{D})$ and so $\{f_k\}$ converges to f because of (4.14); obviously, $\|f\|_{H^-_{SL}(\mathcal{D})} = 1$. In particular, we deduce that the sequence $C(L_0 + \delta L - \lambda_k, \iota'\iota)^{-1})\, f_k$ converges to $-f$ whence

$$f = -Cu \tag{4.15}$$

with $u \neq 0$.

If the sequence $\{\lambda_k\}$ is bounded in $\mathbb{C}$, then using the weak compactness principle and passing to a subsequence, if necessary, we may assume that $\{\lambda_k\}$ converges to $\lambda_0 \in \mathbb{C}$ which satisfies (4.10) and $|\lambda| \geq M_\varepsilon$. Since

$$\begin{aligned}
&(L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1} f_k - (L_0 + \delta L - \lambda_0\, \iota'\iota)^{-1} f \\
&= \ (L_0{+}\delta L{-}\lambda_j\, \iota'\iota)^{-1})(f_k{-}f) + \Big((L_0{+}\delta L{-}\lambda_k\, \iota'\iota)^{-1} - (L_0{+}\delta L{-}\lambda_0\, \iota'\iota)^{-1}\Big) f
\end{aligned}$$

and

$$\begin{aligned}
&\| \Big((L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1} - (L_0 + \delta L - \lambda_0\, \iota'\iota)^{-1}\Big) f \|_{H^-_{SL}(\mathcal{D})} \\
&\leq \ |\lambda_k{-}\lambda_0|\, \|(L_0{+}\delta L{-}\lambda_k\, \iota'\iota)^{-1}\|\, \|(L_0{+}\delta L{-}\lambda_0\, \iota'\iota)^{-1}\|\, \|f\|_{H^-_{SL}(\mathcal{D})},
\end{aligned}$$

estimate (4.8) implies that in this case the sequence $\{(L_0 + \delta L - \lambda_k\, \iota'\iota)^{-1} f_k\}$ converges to $(L_0 + \delta L - \lambda_0\, \iota'\iota)^{-1} f$, and so

$$\Big(I + C(L_0 + \delta L - \lambda_0\, \iota'\iota)^{-1}\Big) f = 0$$

because of (4.14). But λ_0 satisfies (4.10) and $|\lambda| \geq M_\varepsilon$, and hence the injectivity of the operator $I + C(L_0 + \delta L - \lambda_0\, \iota'\iota)^{-1}$ established above yields $f = 0$. This contradicts $\|f\| = 1$.

If $\{\lambda_k\}$ is unbounded in $\mathbb{C}$ we can repeat the arguments above. Indeed, then $f_k = (L_0 + \delta L - \lambda_k\, \iota'\iota)u_k$ and on passing to the weak limit with respect to $k \to \infty$ we get

$$f = (L_0 + \delta L)u - \lim_{k\to\infty} \lambda_k\, \iota'\iota\, u_k.$$

As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\, u_k\}$ converges to $\iota'\iota\, u$ in the space $H^-_{SL}(\mathcal{D})$. Moreover, $\iota'\iota\, u \neq 0$ because of (4.15) and the injectivity of $\iota'\iota$. This shows that the weak limit

$$\lim_{n\to\infty} \lambda_k\, \iota'\iota\, u_k = (L_0 + \delta L)u - f$$

fails to exist if $\{\lambda_k\}$ is unbounded in $\mathbb{C}$, a contradiction. Therefore, $N_\varepsilon > 0$ and for all $\lambda \in \mathbb{C}$ satisfying (4.10) and $|\lambda| \geq M_\varepsilon$ we obtain

$$\| \Big(I + C(L_0 + \delta L - \lambda\, \iota'\iota)^{-1}\Big)^{-1} \|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \leq 1/N_\varepsilon. \tag{4.16}$$

From estimates (4.8), (4.16) and formula (4.11) it follows that, given any $\lambda \in \mathbb{C}$ satisfying (4.10) and $|\lambda| \geq M_\varepsilon$, the resolvent $\mathcal{R}(\lambda; T)$ exists and

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H^-_{SL}(\mathcal{D}))} \leq \mathrm{const}\,(\varepsilon)\,|\lambda|^{-1}.$$

As C is compact, there are only finitely many $\lambda \in \mathbb{C}$ with $|\lambda| < N_\varepsilon$, such that the operator $(I + C(L_0 - \lambda\,\iota'\iota))$ is not injective. Therefore, it follows from formula (4.11) that all eigenvalues of the operator T corresponding to $L_0 + \delta L + C$ (except for a finite number) belong to the corner $|\arg\lambda| < \arcsin\|\delta L\| + \varepsilon$. Finally, since $\varepsilon > 0$ is arbitrary, all rays (4.7) are rays of minimal growth. By the hypothesis of the theorem, the angles between the pairs of neighbouring rays $\arg\lambda = \vartheta$ satisfying (4.7) are less than $2\pi s/n$, and so the statement of the theorem follows from Theorem 4.2. □

5. A non-coercive problem

To the best of our knowledge the completeness of root functions has been studied for elliptic boundary value problems, i.e., for those satisfying the Shapiro-Lopatinskii conditions in domains with smooth boundary or those satisfying coercive estimate (2.5) in domains with Lipschitz boundary. In this section we consider an example where these conditions are violated.

Let the complex structure in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ be given by $z_j = x_j + \sqrt{-1}x_{n+j}$ with $j = 1, \dots, n$ and $\bar\partial$ stand for the Cauchy-Riemann operator corresponding to this structure, i.e., the column of n complex derivatives

$$\frac{\partial}{\partial \bar z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial x_{n+j}}\Big).$$

The formal adjoint $\bar\partial^*$ of $\bar\partial$ with respect to the usual Hermitian structure in the space $L^2(\mathbb{C}^n)$ is the line of n operators

$$-\frac{1}{2}\Big(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial x_{n+j}}\Big) =: -\frac{\partial}{\partial z_j}.$$

An easy computation shows that $\bar\partial^*\bar\partial$ just amounts to the $-1/4$ multiple of the Laplace operator

$$\Delta_{2n} = \sum_{j=1}^{2n}\Big(\frac{\partial}{\partial x_j}\Big)^2$$

in $\mathbb{R}^{2n}$.

As A we take

$$A = -\Delta_{2n} + \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial \bar z_j} + a_0(z),$$

where $a_1(z), \dots, a_n(z)$ and $a_0(z)$ are bounded functions in $\mathcal{D}$, $\mathcal{D}$ being a bounded domain with Lipschitz boundary in $\mathbb{C}^n$. The Hermitian matrix

$$\big(a_{i,j}\big)_{\substack{i=1,\dots,2n\\ j=1,\dots,2n}}$$

is chosen to be

$$\begin{pmatrix} E_n & \sqrt{-1}E_n \\ -\sqrt{-1}E_n & E_n \end{pmatrix}$$

where E_n is the unity $(n \times n)$-matrix. Obviously, $\overline{a_{i,j}} = a_{j,i}$ for all $i, j = 1, \ldots, 2n$ and the corresponding conormal derivative is

$$\partial_c = 2 \sum_{j=1}^{n} (\nu_j - \sqrt{-1}\nu_{n+j}) \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial \nu} + \sqrt{-1} \sum_{j=1}^{n} \Big(\nu_j \frac{\partial}{\partial x_{n+j}} - \nu_{n+j} \frac{\partial}{\partial x_j} \Big),$$

which is known as complex normal derivative $\bar{\partial}_\nu$ at the boundary of $\mathcal{D}$.

Consider the following boundary value problem. Given a distribution f in $\mathcal{D}$, find a distribution u in $\mathcal{D}$ satisfying

$$\left\{ \begin{array}{rcll} \Big(- \varDelta_{2n} + \displaystyle\sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial \bar{z}_j} + a_0(z) \Big) u & = & f & \text{in} \quad \mathcal{D}, \\ \bar{\partial}_\nu u + b_0(z) u & = & 0 & \text{on} \quad \partial\mathcal{D}. \end{array} \right. \tag{5.1}$$

In this case S is empty, $t = 0$ and $b_1 = 1$. Set $a_{0,0}(z) \equiv 1$ in $\mathcal{D}$ and $b_{0,0}$ to be any positive constant function on the boundary. Then the corresponding Hermitian form $(\cdot, \cdot)_{SL}$ is

$$(u, v)_{SL} = (u, v)_{L^2(\mathcal{D})} + (2\bar{\partial} u, 2\bar{\partial} v)_{L^2(\mathcal{D})} + b_{0,0}\, (u, v)_{L^2(\partial\mathcal{D})}$$

and the space $H_{SL}(\mathcal{D})$ is defined to be the completion of $H^1(\mathcal{D})$ with respect to the norm $\|u\|_{SL} = \sqrt{(u, u)_{SL}}$.

Remark 5.1. By Theorem 2.5, the space $H_{SL}(\mathcal{D})$ is continuously embedded into $H^{1/2-\varepsilon}(\mathcal{D})$ with any $\varepsilon > 0$. However, there is no continuous embedding $H_{SL}(\mathcal{D}) \hookrightarrow H^{1/2+\varepsilon}(\mathcal{D})$ with $\varepsilon > 0$. Indeed, if $\mathcal{D}$ is the unit disc in $\mathbb{C}$ then a direct calculation using Lemma 1.4 of [Shl96] shows that the series

$$u_\varepsilon(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)^{(1+\varepsilon)/2}},$$

where $\varepsilon > 0$, converges in $H_{SL}(\mathcal{D})$ but diverges in $H^{1/2+\varepsilon}(\mathcal{D})$. This means that the coercive estimate (2.5) does not hold for problem (5.1). Besides, as the monomials z^k are L^2-orthogonal on the circles $|z| = r$, we see that in this case the term induced by a perturbation $\delta b_0 \in \mathbb{C}$ fails to be a compact operator from $H_{SL}(\mathcal{D})$ to $H^-_{SL}(\mathcal{D})$ (cf. [PS13]).

As $t = 0$ and $b_1 \equiv 1$, we deduce that estimate (2.19) is valid, provided that the functions a_j and a_0 are of class $L^\infty(\mathcal{D})$ and $b_0 \in L^\infty(\partial\mathcal{D})$. The operator L_0 corresponds to

$$\left\{ \begin{array}{rcll} -\varDelta_{2n} u + u & = & f & \text{in} \quad \mathcal{D}, \\ \bar{\partial}_\nu u + b_{0,0}\, u & = & 0 & \text{on} \quad \partial\mathcal{D}. \end{array} \right. \tag{5.2}$$

Theorem 5.2. *The inverse L_0^{-1} of the operator L_0 related to (5.2) induces compact positive selfadjoint operators*

$$\begin{array}{rclcrcl} Q_1 & = & \iota' \iota\, L_0^{-1} & : & H^-_{SL}(\mathcal{D}) & \to & H^-_{SL}(\mathcal{D}), \\ Q_2 & = & \iota\, L_0^{-1}\, \iota' & : & L^2(\mathcal{D}) & \to & L^2(\mathcal{D}), \\ Q_3 & = & L_0^{-1}\, \iota' \iota & : & H_{SL}(\mathcal{D}) & \to & H_{SL}(\mathcal{D}) \end{array}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are positive and there are orthonormal bases in $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$ consisting of the eigenvectors.

Proof. For the proof it suffices to combine Lemma 3.1 and Theorem 2.5. $\square$

Corollary 3.5 actually shows that the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n+\varepsilon}$ for any $\varepsilon > 0$, and so their orders are finite.

Theorem 5.3. *Let $t = 0$ and $\delta b_0 = 0$. Then operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ related to problem (5.1) is Fredholm. Moreover, the system of root functions of the corresponding closed operator T is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. For any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg\lambda| < \varepsilon$.*

Proof. Indeed, the operator

$$\delta L = L - L_0 : \ H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$$

maps $H_{SL}(\mathcal{D})$ continuously to $L^2(\mathcal{D})$, and hence it is compact. Thus, the statement follows from Theorem 3.8. $\square$

Corollary 5.4. *Let $t = 0$ and $\delta b_0 = 0$. If $\|a_j\|_{L^\infty(\mathcal{D})} < 1$, for $1 \leq j \leq n$, and $\|a_0 - 1\|_{L^\infty(\mathcal{D})} < 1$, then the operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ corresponding to problem (5.1) is continuously invertible. Moreover, the system of root functions of the compact operator $\iota'\iota L^{-1}$ in $H^-_{SL}(\mathcal{D})$ is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. For any $\varepsilon > 0$, all eigenvalues of $\iota'\iota L^{-1}$ (except for a finite number) belong to the corner $|\arg\lambda| < \varepsilon$.*

Proof. We just recall that estimate (2.19) in the particular case under consideration becomes explicitly

$$\Big|\Big(\sum_{j=1}^{n} a_j(z)\frac{\partial u}{\partial \bar{z}_j} + (a_0 - 1)\, u, v\Big)_{L^2(\mathcal{D})}\Big| \leq c\, \|u\|_{SL}\|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D})$, with c being the maximal of the numbers $\|a_j\|_{L^\infty(\mathcal{D})}$, where $j = 1, \dots, n$, and $\|a_0 - 1\|_{L^\infty(\mathcal{D})}$. $\square$

Corollary 5.5. *If $t = 0$ and the norms $\|a_j\|_{L^\infty(\mathcal{D})}$, $\|a_0 - 1\|_{L^\infty(\mathcal{D})}$, $\|b_{0,0}^{-1}\,\delta b_0\|_{L^\infty(\partial\mathcal{D})}$ are all majorized by a constant $c < \sin(\pi/2n)$, then the closed operator T in $H^-_{SL}(\mathcal{D})$ corresponding to problem (5.1) is continuously invertible. Moreover, the system of root functions of T is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$, and all eigenvalues of T (except for a finite number) lie in the corner $|\arg\lambda| < \arcsin c$.*

6. The coercive case

We now turn to the coercive case, i.e., we assume that estimate (2.5) is fulfilled. This is obviously the case if all the coefficients $a_{i,j}(z)$ of A are real-valued, which is due to (2.4). According to Lemma 2.4, the space $H_{SL}(\mathcal{D})$ is embedded continuously into $H^1(\mathcal{D}, S)$.

Theorem 6.1. *Let estimate (2.5) and one of the three conditions of Lemma 2.4 be fulfilled. The inverse L_0^{-1} of the operator L_0 induces compact positive selfadjoint operators*

$$\begin{array}{rclcrcl}
Q_1 & = & \iota'\iota L_0^{-1} & : & H^-_{SL}(\mathcal{D}) & \to & H^-_{SL}(\mathcal{D}),\\
Q_2 & = & \iota L_0^{-1}\iota' & : & L^2(\mathcal{D}) & \to & L^2(\mathcal{D}),\\
Q_3 & = & L_0^{-1}\iota'\iota & : & H_{SL}(\mathcal{D}) & \to & H_{SL}(\mathcal{D})
\end{array}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are positive and there are orthonormal bases in $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$ consisting of the eigenvectors.

Proof. For the proof it suffices to combine Lemmas 3.1 and 2.4. □

Corollary 3.5 actually shows that the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$, and so their orders are finite.

Lemma 6.2. *Assume that estimate (2.5) is fulfilled. If one of the three conditions of Lemma 6.2 holds true and $b_{0,0}/b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$, then the norms $\|\cdot\|_{SL}$ and $\|\cdot\|_{H^1(\mathcal{D})}$ are equivalent and the spaces $H_{SL}(\mathcal{D})$ and $H^1(\mathcal{D}, S)$ coincide as topological ones.*

Proof. The proof is standard, cf. for instance Section 5.6 in [Mik76, Ch. 3]. □

It should be noted that the case $b_{0,0}/b_1 \notin L^\infty(\partial\mathcal{D} \setminus S)$ is of great interest, too. It can be handled within the framework of mixed problems in weighted spaces with weights which control the behaviour of solutions nearby the interface ∂S, see [Esk73], [Tar06], etc.

Our next task is to describe those perturbations a_j, $\delta b_0/b_1$ and t which preserve the completeness property of root functions of the operator L_0^{-1}. In particular, we will clarify the conditions for (2.19) to hold in the coercive case, and find an estimate for the constant c in this inequality. Write ι_1 for the continuous embedding $H_{SL}(\mathcal{D}) \hookrightarrow H^1(\mathcal{D})$ whose norm depends on S, $a_{0,0}$, $b_{0,0}/b_1$ and the value m from estimate (2.5)).

Lemma 6.3. *Let estimate (2.5) and one of the three conditions of Lemma 2.4 be fulfilled. Then the differential operator*

$$\delta A = \sum_{j=1}^{n} a_j(x)\,\partial_j + \delta a_0\,(x)$$

induces a compact operator $C_1 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$, such that $|(\delta A u, v)_{L^2(\mathcal{D})}| \leq c\,\|u\|_{SL}\|v\|_{SL}$ for all $u, v \in H^1(\mathcal{D}, S)$.

As is seen from the proof, the constant c can be written explicitly through the norms $\|a_j\|_{L^\infty(\mathcal{D})}$, $\|\delta a_0\|_{L^\infty(\mathcal{D})}$ and $\|\iota\|$, $\|\iota_1\|$.

Proof. Obviously, δA maps the space $H^1(\mathcal{D})$ continuously to $L^2(\mathcal{D})$. Lemma 2.4 implies readily that δA maps also the space $H_{SL}(\mathcal{D})$ continuously to $L^2(\mathcal{D})$. We thus get

$$|(\delta A u, v)_{L^2(\mathcal{D})}| \leq c'\,\|u\|_{H^1(\mathcal{D})}\|v\|_{L^2(\mathcal{D})}$$

for all $u, v \in H_{SL}(\mathcal{D})$, where c' is any constant majorizing the norms $\|a_j\|_{L^\infty(\mathcal{D})}$ and $\|\delta a_0\|_{L^\infty(\mathcal{D})}$. By Lemma 2.2, the embedding $\iota' : L^2(\mathcal{D}) \hookrightarrow H^-_{SL}(\mathcal{D})$ is compact, and so δA induces a compact operator $C_1 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$. The desired estimate is now a consequence of the very definitions of ι and ι_1. □

Lemma 6.4. *Let estimate (2.5) and one of the three conditions of Lemma 2.4 be fulfilled. If $b_1^{-1}\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$ then*

$$|((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \leq c\,\|u\|_{SL}\,\|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D}, S)$, where the constant c is independent of u, v. Moreover, the term $((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}$ in the weak formulation (2.21) of problem (2.2) induces a compact linear operator $C_2 : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$.

Proof. Since $b_1^{-1}\delta b_0$ is a bounded function on $\partial\mathcal{D} \setminus S$, it follows that

$$|((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \quad \le \quad \|b_1^{-1}\delta b_0\|_{L^\infty(\partial\mathcal{D}\setminus S)} \|u\|_{L^2(\partial\mathcal{D})}\|v\|_{L^2(\partial\mathcal{D})} \tag{6.1}$$

for all $u, v \in H^1(\mathcal{D}, S)$. Furthermore, by (6.1), Lemma 2.4 and the trace theorem, we obtain

$$|((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \le c_1 \|u\|_{L^2(\partial\mathcal{D})}\|v\|_{SL} \le c \|u\|_{SL}\|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D}, S)$, as desired. Hence, the summand $((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}$ in the weak formulation (2.21) of problem (2.2) induces a bounded linear operator

$$C_2 : H_{SL}(\mathcal{D}) \to H_{SL}^-(\mathcal{D})$$

satisfying

$$\|C_2 u\|_{H_{SL}^-(\mathcal{D})} \le c \|u\|_{L^2(\partial\mathcal{D})}. \tag{6.2}$$

Now, if σ is a bounded set in $H_{SL}(\mathcal{D})$ then, by Lemma 2.4, it is also bounded in $H^1(\mathcal{D})$ and, by the trace theorem, the set $\sigma \restriction_{\partial\mathcal{D}}$ of restrictions of elements of σ to $\partial\mathcal{D}$ is bounded in $H^{1/2}(\partial\mathcal{D})$, too. The Rellich-Kondrashov theorem implies that $\sigma \restriction_{\partial\mathcal{D}}$ is precompact in $H^s(\partial\mathcal{D})$, if $s < 1/2$. Hence, for any sequence $\{u_k\}$ in σ there is a subsequence $\{u_{k_j}\}$ whose restriction to $\partial\mathcal{D}$ is a Cauchy sequence in $L^2(\partial\mathcal{D})$. Using (6.2) we conclude that any sequence $\{u_k\}$ in σ has a subsequence $\{u_{k_j}\}$, such that $\{C_2 u_{k_j}\}$ is a Cauchy sequence in $H_{SL}^-(\mathcal{D})$. Thus, the operator $C_2 : H_{SL}(\mathcal{D}) \to H_{SL}^-(\mathcal{D})$ is compact. This establishes the lemma. □

Lemma 6.5. *If $b_{0,0}^{-1}\,\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$ then*

$$|((b_1^{-1}\,\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \le \|b_{0,0}^{-1}\,\delta b_0\|_{L^\infty(\mathcal{D})} \|u\|_{SL}\,\|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D}, S)$, the constant c being independent of u and v.

Proof. Since $b_{0,0}^{-1}\delta b_0$ is a bounded function on $\partial\mathcal{D} \setminus S$, it follows that

$$\begin{aligned} |((b_1^{-1}\,\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| &= |(b_1^{-1} b_{0,0}\,(b_{0,0}^{-1}\delta b_0))u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \\ &\le c\,\|\sqrt{b_1^{-1} b_{0,0}}u\|_{L^2(\partial\mathcal{D}\setminus S)}\,\|\sqrt{b_1^{-1} b_{0,0}}v\|_{L^2(\partial\mathcal{D}\setminus S)} \\ &\le c\,\|u\|_{SL}\|v\|_{SL} \end{aligned}$$

for all $u, v \in H^1(\mathcal{D}, S)$, where $c = \|b_{0,0}^{-1}\delta b_0\|_{L^\infty(\mathcal{D})}$, as desired. □

Let $t^1(x), \dots, t^{n-1}(x)$ be a basis of tangential vectors of the boundary surface at a point $x \in \partial\mathcal{D}$. Then we can write

$$\partial_t = \sum_{j=1}^{n-1} t_j(x)\,\partial_{t^j}$$

where $t_1, \dots, t_{n-1}$ are bounded functions on the boundary vanishing on S.

Lemma 6.6. *Let (2.5) be fulfilled and S be an open subset on $\partial\mathcal{D}$. If $t_j/b_1 \in C^{0,\lambda}(\partial\mathcal{D} \setminus S)$, for $1 \le j \le n-1$, with $\lambda > 1/2$, then there is a constant $c > 0$ such that*

$$\Big| \int_{\partial\mathcal{D}\setminus S} b_1^{-1}\partial_t u\,\overline{v}\,ds \Big| \le c\,\|u\|_{SL}\|v\|_{SL} \tag{6.3}$$

for all $u, v \in H^1(\mathcal{D}, S)$.

Proof. We first note that $\partial\mathcal{D}$ is a closed compact manifold and ∂_{t^j} are first order differential operators with bounded coefficients on $\partial\mathcal{D}$. These operators map $H^{1/2}(\partial\mathcal{D})$ continuously to $H^{-1/2}(\partial\mathcal{D}) = (H^{1/2}(\partial\mathcal{D}))'$.

Recall that every Hölder continuous function $f_0 \in C^{0,\lambda}(K)$ on a compact set $K \subset \mathbb{R}^n$, with $0 < \lambda < 1$, extends to a Hölder continuous function $f \in C^{0,\lambda}(\mathbb{R}^n)$ on all of $\mathbb{R}^n$, such that $\|f\|_{C^{0,\lambda}(\mathbb{R}^n)} \leq \|f_0\|_{C^{0,\lambda}(K)}$.

As S is open on $\partial\mathcal{D}$, the set $\partial\mathcal{D} \setminus S$ is a compact. Since $t_j/b_1 \in C^{0,\lambda}(\partial\mathcal{D} \setminus S)$, where $1/2 < \lambda \leq 1$, we see that for each $1 \leq j \leq n-1$ there is an extension $f_j \in C^{0,\lambda}(\partial\mathcal{D})$ which satisfies

$$\|f_j\|_{C^{0,\lambda}(\partial\mathcal{D})} \leq \|t_j/b_1\|_{C^{0,\lambda}(\partial\mathcal{D}\setminus S)}.$$

On the other hand, we have $f_j v = e^+(t_j/b_1)v$ for each $1 \leq j \leq n-1$ and $v \in C^1(\overline{\mathcal{D}}, S)$, where e^+ is the extenson operator of functions on $\partial\mathcal{D} \setminus S$ by zero to S. Therefore, there exists a constant $c > 0$ such that

$$\|(t_j/b_1)v\|_{H^{1/2}(\partial\mathcal{D})} = \|f_j v\|_{H^{1/2}(\partial\mathcal{D})} \leq c\,\|v\|_{H^{1/2}(\partial\mathcal{D})} \tag{6.4}$$

for all $v \in C^1(\overline{\mathcal{D}}, S)$, because the multiplication by functions f_j of Hölder class $C^{0,\lambda}(\partial\mathcal{D})$ with $1/2 < \lambda \leq 1$ is a bounded linear operator m_{f_j} in $H^{1/2}(\partial\mathcal{D})$, see [Slo58, §3]. On applying (6.4) we see that

$$\begin{aligned}
\Big|\int_{\partial\mathcal{D}\setminus S} b_1^{-1}\partial_t u\,\overline{v}\,ds\Big| &\leq c\sum_{j=1}^{n-1}\|\partial_{t^j}u\|_{H^{-1/2}(\partial\mathcal{D})}\|v\|_{H^{1/2}(\partial\mathcal{D})}\\
&\leq c\,\|u\|_{H^{1/2}(\partial\mathcal{D})}\|v\|_{H^{1/2}(\partial\mathcal{D})}\\
&\leq c\,\|u\|_{H^1(\mathcal{D})}\|v\|_{H^1(\mathcal{D})}\\
&\leq c\,\|u\|_{SL}\|v\|_{SL}
\end{aligned}$$

for all $u, v \in C^1(\overline{\mathcal{D}}, S)$, where the constant c does not depend on u and v and may be diverse in different applications. Finally, (6.3) is fulfilled because $C^1(\overline{\mathcal{D}}, S)$ is dense in $H^1(\mathcal{D}, S)$. □

Thus, Lemmas 6.3, 6.4, 6.5 and 6.6 and Remark describe the behavior of typical perturbations δL in the coercive case. In particular, they give us rather sharp estimates for the constant c of (2.19).

Theorem 6.7. *Suppose that estimate (2.5) and one of the three conditions of Lemma 2.4 are fulfilled. If $t = 0$ and $b_1^{-1}\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$, then the operator $L : H_{SL}(\mathcal{D}) \to H_{SL}^-(\mathcal{D})$ related to Problem (2.21) is Fredholm. Moreover, the system of root functions of the corresponding closed operator $T : H_{SL}^-(\mathcal{D}) \to H_{SL}^-(\mathcal{D})$ is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$, and, for arbitrary $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg\lambda| < \varepsilon$ in $\mathbb{C}$.*

Proof. For the proof it suffices to apply Theorem 3.6 combined with Lemmas 2.4, 6.3 and 6.4, for the linear operator L is Fredholm if $(L - L_0)$ is compact. □

Set

$$C := \Big(\sum_{j=1}^{n-1}\|\partial_{t^j}\|\,\|m_{f_j}\|\Big)\|\tau\|^2\|\iota_1\|^2,$$

where $\tau : H^1(\mathcal{D}) \to H^{1/2}(\partial\mathcal{D})$ is the trace operator. This constant appears persistently in the last two theorems.

Theorem 6.8. *Let estimate (2.5) and one of the three conditions of Lemma 2.4 be fulfilled. Suppose that $b_1^{-1}\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$, $t_j/b_1 \in C^{0,\lambda}(\partial\mathcal{D} \setminus S)$, for $1 \le j \le n-1$, with $\lambda > 1/2$, and the constant C does not exceed $\sin(\pi/n)$. Then the operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ related to problem (2.21) is Fredholm. Moreover, the system of root functions of the corresponding closed operator T in $H^-_{SL}(\mathcal{D})$ is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$, and, for any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) lie in the corner $|\arg\lambda| < \arcsin C + \varepsilon$ in the complex plane.*

Proof. This is an immediate consequence of Theorem 4.5 and Lemmas 2.4, 2.7, 6.3, 6.4 and 6.6. □

Theorem 6.9. *Let estimate (2.5) and one of the three conditions of Lemma 2.4 be fulfilled. If $b_{0,0}^{-1}\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$, $t_j/b_1 \in C^{0,\lambda}(\partial\mathcal{D} \setminus S)$, for $1 \le j \le n-1$, with $\lambda > 1/2$, and*

$$\|b_{0,0}^{-1}\,\delta b_0\|_{L^\infty(\partial\mathcal{D}\setminus S)} + C \le \sin(\pi/n),$$

then the operator $L : H_{SL}(\mathcal{D}) \to H^-_{SL}(\mathcal{D})$ related to problem (2.21) is Fredholm. Moreover, the system of root functions of the corresponding closed operator T in $H^-_{SL}(\mathcal{D})$ is complete in the spaces $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$, and, for any $\varepsilon > 0$, all but a finite number of eigenvalues of T lie in the corner

$$|\arg\lambda| < \arcsin\Big(\|b_{0,0}^{-1}\,\delta b_0\|_{L^\infty(\partial\mathcal{D}\setminus S)} + C\Big) + \varepsilon.$$

Proof. This assertion follows immediately from Theorem 4.5, Lemmas 2.4, 2.7, 6.3, 6.5 and 6.6 as well. □

7. Zaremba type problems

As but one example of boundary value problems satisfying the coercive estimate (2.5) we consider the mixed problem

$$\left\{\begin{array}{rcll} -\Delta_n u + \displaystyle\sum_{j=1}^n a_j(x)\partial_j u + a_0(x)u & = & f & \text{in} \quad \mathcal{D}, \\ u & = & 0 & \text{on} \quad S, \\ \partial_\vartheta u & = & 0 & \text{on} \quad \partial\mathcal{D} \setminus S \end{array}\right. \tag{7.1}$$

for a real-valued function u, where Δ_n is the Laplace operator in $\mathbb{R}^n$, the coefficients $a_1, \dots, a_n$ and a_0 are assumed to be bounded functions in $\mathcal{D}$, and $\partial_\vartheta := \partial_\nu + \varepsilon\partial_t$ with $\varepsilon \in \mathbb{C}$ and t a tangential vector field on $\partial\mathcal{D}$. By a theorem of Rademacher, $t(x)$ is defined almost everywhere at the boundary and its coefficients are bounded functions on $\partial\mathcal{D}$. We assume that t vanishes on S and is of Hölder class $C^{0,\lambda}$ with $\lambda > 1/2$ in $\partial\mathcal{D} \setminus S$, cf. [Zar10].

In this case $a_{i,j} = \delta_{i,j}$, $b_0 = \chi_S$ is the characteristic function of the boundary set S, and $b_1 = \chi_{\partial\mathcal{D}\setminus S}$ is that of $\partial\mathcal{D} \setminus S$.

From results of the previous section it follows that the root functions related to problem (7.1) in the space $H_{SL}(\mathcal{D})$ are complete in $H^-_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$ for all ε of sufficiently small modulus.

We finish the paper by showing a second order elliptic differential operator in $\mathbb{R}^2$ for which no Zaremba-type problem is Fredholm. The idea is traced back to a familiar example of A.V. Bitsadze (1948).

Example 7.1. Let $A = \bar{\partial}^2$ be the square of the Cauchy-Riemann operator in the plane of complex variable z. We choose $\mathcal{D}$ to be the upper half-disk of radius 1, i.e., the set of all $z \in \mathbb{C}$ satisfying $|z| < 1$ and $\Im z > 0$. As S we take the upper half-circle, i.e., the part of $\partial\mathcal{D}$ lying in the upper half-plane. Consider the function sequence

$$u_k(z) = (|z|^2 - 1)\, \frac{\sin(kz)}{k^s},$$

for $k = 1, 2, \ldots$, where s is a fixed positive number. Each function u_k vanishes on S. Moreover, for any differential operator B of order $< s$ with bounded coefficients, the sequence $\{Bu_k\}$ converges to zero uniformly on $\partial\mathcal{D} \setminus S = [-1, 1]$. Since $|u_k(z)| \to \infty$ for all $z \in \mathcal{D}$, we deduce that no reasonable setting of Zaremba-type problem is possible.

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