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On Iterations of the Green Integrals and their Applications to Elliptic Differential Complexes

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Abstract. Convergence of iterations of special Green integrals for overdetermined elliptic linear partial differential operators P of order $p \geq 1$ is proved. Using this result we obtain necessary and sufficient conditions for the solvability in Sobolev space $W^{p,2}(D)$ of the equation $Pu = f$ and, as a corollary, necessary and sufficient conditions for the vanishing of the first cohomology group of elliptic differential complexes. Also a criterion for the solvability of a P-Neumann problem for elliptic differential operators is proved.

1. Introduction

The validity of the Poincarè lemma, i.e. local acyclicity, for elliptic complexes of linear partial differential operators with smooth coefficients is a long standing problem of the theory of overdetermined systems.

Although we are still not able to settle this question, we succeed in this paper in proving a representation formula for solutions of the equation

$$Pu = f \tag{1.1}$$

for an operator P with injective symbol whenever they exist.

This representation involves the sum of a series whose terms are iterations of integro-differential operators, while solvability of (1.1) is equivalent to the convergence of the series together with the orthogonality to a harmonic space (the last one is a trivial necessary condition).

For the Dolbeault complex, these integro-differential operators are related to the Martinelli-Bochner integral. In this case, results similar to ours were obtained by A.V. Romanov [11].

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Although this example shows that in general we should expect a loss of global SOBOLEV regularity for the solutions of (1.1), the case where (1.1) can be solved without losing global regularity has interesting applications to a variational non elliptic boundary value problem, that we discuss and illustrate by the examples at the end of the paper.

Let us describe in a more precise way the contents of this paper.

Let X be an open set in $\mathbb{R}^n$ $(n \geq 1)$ and P be an elliptic $(l \times k)$-matrix of partial differential operators of order $p \geq 1$ with C^∞ coefficients in X. We are interested in the solvability of the equation (1.1) in a relatively compact domain D in X. Our article is based on the following simple but useful observation.

Let H be a linear topological vector space of (vector-valued) functions defined in D and let us assume that for every $u \in H$ the following formula holds true:

$$u = \Pi_1 u + \Pi_2 P u \tag{1.2}$$

where $\Pi_1, \Pi_2 P : H \to H$ and Π_1 is a projection from H to the subspace $\{u \in H : Pu = 0 \ \mathit{in}\ D\}$ of H. Then one can hope that, under reasonable conditions, the element $\Pi_2 f$ defines a solution of the equation $Pu = f$ in D.

For instance, such an approach was successfully tested on the CAUCHY- RIEMANN system $\overline{\partial}$ in $\mathbb{C}^n$ $(n > 1)$ and formulae of the type (1.2) were obtained in [1], [4] (see also [3]) by the method of integral representations. The construction of formula 1.2 by the method of integral representations demands the construction of the special holomorphic kernels for the integral Π_1, essentially depending on the domain D.

In the present paper we use another idea which was also first introduced in complex analysis.

In 1978 two papers of A.V. ROMANOV devoted to the iterations of the MARTINELLI-BOCHNER integral were published (see [10], [11]. In particular, in [11] the following result was obtained.

Theorem 1.1. (A.V. ROMANOV.) *Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected boundary ∂D of class C^1, and let M be the* MARTINELLI-BOCHNER *integral (on ∂D) defined on the* SOBOLEV *space $W^{1,2}(D)$. Then in the strong operator topology in $W^{1,2}(D)$*

$$\lim_{\nu \to \infty} M^\nu = \Pi_1$$

where Π_1 is a projection from $W^{1,2}(D)$ onto the closed subspace of holomorphic $W^{1,2}(D)$ -functions.

Using this theorem ROMANOV (see [11]) obtained a multi-dimensional analogue of the CAUCHY-GREEN formula in the plane (see, for example, [4]), i.e. a formula of the type (1.2), and, as consequence, an explicit formula for a solution $u \in W^{1,2}(D)$ of the equation $\overline{\partial} u = f$ where D is a pseudo-convex domain with a smooth (infinitely differentiable) boundary, and f is a $\overline{\partial}$ -closed (0,1)-form with coefficients in $W^{1,2}(D)$.

The GREEN integrals (see, for example, [16]) associated to systems of linear differential equations with injective symbols are natural analogues of the MARTINELLI-BOCHNER integral. Within this more general context in the present paper the possibilities to prove a similar result to the theorem of ROMANOV and its applications are discussed.

The plan of the paper is the following.

Section 1 is devoted to Green operators, Green integrals, and integral representations for solutions of systems with injective symbols.

The scheme of the proof of the theorem on iterations for the Green integrals is described in Section 2. This scheme is a variation of the original proof by A.V. Romanov [11]. Also some immediate consequences of this theorem are shown in this section.

In Section 3 the theorem on iterations is established for the Green integrals (associated to differential operators with injective symbols) which are constructed by means of special left fundamental solutions (Green functions).

Using results of Section 4, in Section 5 we obtain solvability conditions for equation (1.1) in the case where the operator P is overdetermined.

In Section 6 we study the first Sobolev cohomology group of elliptic differential complexes. In particular we obtain criterions for its vanishing.

In Section 7 we obtain necessary and sufficient conditions for the solvability in the Sobolev spaces of a P-Neumann problem for elliptic differential operators.

After discussing in Section 8 some examples, we consider in Section 9 some applications of the Theorem on iterations to the Cauchy and Dirichlet problems.

Sections 7 and 9 were inspired by results of Kytmanov [7] for the multi-dimensional Cauchy-Riemann system.

2. Green integrals and Green operators

Let $X \subset \mathbb{R}^n$ be an open set, $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ be (trivial) vector bundles over X, and $do_p(E \to F)$ be the vector space of smooth linear partial differential operators of order $\leq p$ between the vector bundles E and F. Throughout this article we will mostly use the letters v, u for sections of E, and the letters f, g for sections of F. Sections of E and F of a class $\mathfrak{C}$ on an open set $\sigma \subset X$ can be interpreted as columns of complex valued functions from $\mathfrak{C}(\sigma)$, that is, $\mathfrak{C}(E_{|\sigma}) \cong [\mathfrak{C}(\sigma)]^k$, and similarly for F. Then $P \in do_p(E \to F)$ is an $(l \times k)$ matrix of scalar linear partial differential operators, i.e. we have

$$P(x, D) = \sum_{|\alpha| \leq p} P_\alpha(x) D^\alpha$$

where $P_\alpha(x)$ are $(l \times k)$-matrices of smooth functions on X.

Let E^* be the dual bundle of E, and let $(.,.)_x$ be a Hermitian metric on E. Then $*_E : E \to E^*$ is defined by $< *_E v, u >_x = (u, v)_x$ (where u, v are sections of E and $< .,. >_x$ is the natural pairing of E and E^*). Let Λ^q be the bundle of complex valued exterior forms of degree q ($q = 1, 2, \ldots$) over X, and dx the usual volume form on X.

We denote by ${}^tP \in do_p(F^* \to E^*)$ the transposed operator, and by $P^* = *_E^{-1}({}^tP)*_F \in do_p(F \to E)$ the (formal) adjoint operator of $P \in do_p(E \to F)$.

Definition 2.1. A differential bilinear operator $G_P(\cdot,\cdot \in do_{p-1}((F^*,E) \to \Lambda^{n-1})$ is said to be a Green operator for $P \in do_p(E \to F)$ if the following formula holds:

$$dG_P(g,v) =< g, Pv >_x dx- < {}^tPg, v >_x dx \ (g \in C^\infty(F^*), v \in C^\infty(E)).$$

Green operators always exist for every differential operator P (see [16], p.82). For instance, a Green operator G_P can be written in the form

$$G_P(g,v) = \sum_{|\beta+\gamma+1_j|\le p}{}'(-1)^\beta D^\beta(gP_{\beta+\gamma+1_j})D^\gamma v * dx_j \tag{2.1}$$

where $\sum'$ indicate that an order has been selected with respect to the multi-indexes $\beta, \gamma, 1_j$, and $*$ is the Hodge operator (see [16], p.82).

For the purposes of this paper it is more convenient to write Green operators in another form.

Let D be a relatively compact domain in X with smooth boundary, and let U be a neighbourhood of ∂D in X, and $F_j = U \times C^k$ $(0 \le j \le r < \infty)$ be (trivial) vector bundles over U.

Definition 2.2. A system $\{B_j\}_{j=0}^r$ of differential operators $B_j \in do_{b_j}(E_{|U} \to F_j)$ is said to be a Dirichlet system of order r on ∂D if 1) $0 \le b_j \le r$; 2) $b_j \ne b_i$ for $j \ne i$; 3) $rank_{\mathbb{C}}\, \sigma(B_j)(y, d\rho) = k$ $(0 \le j \le r), y \in U$, where $\sigma(B_j)$ is the principal symbol of the operator B_j, and ρ belongs to the class of functions defining the domain D $(D = \{x \in X : \rho(x) < 0,\ d\rho \ne 0 \ in \ U\})$.

The following lemma was proved in [17] (p.280, Lemma 28.3).

Lemma 2.3. *Suppose that the boundary ∂D of D is non characteristic for $P \in do_p(E \to F)$ $(l \ge k)$. Then, given a* Dirichlet *system $\{B_j\}_{j=0}^{p-1}$, one can find a neighbourhood U of ∂D, and a* Green *operator G_P such that*

$$G_P(g,v) = \sum_{j=0}^{p-1} < C_j g, B_j v >_x ds \ + \ \frac{d\rho}{|d\rho|} \wedge G_\nu(g,v) \ (g \in C^\infty(F^*_{|U}), v \in C^\infty(E_{|U}))$$

where $\{C_j\}_{j=0}^{p-1}$ is a Dirichlet *system of order $(p-1)$ on ∂D with $C_j \in do_{p-b_j-1}(F^*_{|U} \to F^*_j)$ $(0 \le j \le p-1)$ and $G_\nu \in do_{p-1}((F^*,E)_{|U} \to \Lambda^{n-2})$.*

Without loss of a generality we assume that $b_j = j$. For example, we can set $B_j = I_k \frac{\partial^j}{\partial n^j}$, where $\frac{\partial^j}{\partial n^j}$ is the j-th normal derivative with respect to ∂D and I_k is the unit $(k \times k)$-matrix.

Using Green operators one obtains integral representations for solutions of the system $Pu = 0$.

We say that the linear partial differential operator $P \in do_p(E \to F)$ is elliptic if its principal symbol

$$\sigma(P)(x,\zeta) = \sum_{\alpha=p} P_\alpha(x)\zeta^\alpha : \mathbb{C}^k \to \mathbb{C}^l$$

is injective for every $x \in X$ and $\zeta \in \mathbb{R}^n \setminus \{0\}$. In particular $l \ge k$; we say that P is determined elliptic if $l = k$ and overdetermined elliptic if $l > k$. Every determined elliptic operator with smooth coefficients has locally a bilateral (i.e. left and right) fundamental solution, and hence every overdetermined elliptic operator with smooth coefficients has locally a left fundamental solution. If the coefficients of the operator P are real analytic, there exist global fundamental solutions of the operator P on X (cf., for example, [16], §8). From now on we will assume throughout the paper that the operator P is elliptic.

We will denote by $W^{m,2}(E_{|D})$ the Sobolev space of distribution sections of E over D having weak derivatives in $L^2(E_{|D})$ up to order m and by $S_P^{m,2}(D)$ the closed linear subspace of $W^{m,2}(E_{|D})$ of weak solutions of the equation $Pu = 0$ in D.

Theorem 2.4. *Let $\mathcal{L}$ is a (left) fundamental solution of the operator P on X. For every $u \in W^{p,2}(E_{|D})$ the following formula holds:*

$$(2.2) \quad -\int_{\partial D} G_P(\mathcal{L}(x,y), u(y)) + \int_D < \mathcal{L}(x,y), Pu(y) >_y dy = \begin{cases} u(x), & x \in D, \\ 0, & x \in X \setminus \overline{D} \end{cases}$$

Proof. If $u \in C^p(E_{|\overline{D}})$ (that is, u is p times continuously differentiable in a neighbourhood of $\overline{D}$) then (2.2) follows from the Stokes' formula and Definition 1.1. Since the boundary of D is smooth, there exists a sequence of functions $\{u_N\}_{N=1}^{\infty} \in C^p(E_{|\overline{D}})$ approximating u in $W^{p,2}(E_{|D})$. Then for every $N \in \mathbb{N}$

$$-\int_{\partial D} G_P(\mathcal{L}(x,y), u_N(y)) + \int_D < \mathcal{L}(x,y), Pu_N(y) >_y dy = \begin{cases} u_N(x), & x \in D, \\ 0, & x \in X \setminus \overline{D} \end{cases} \quad (2.3)$$

Using the boundedness theorem for pseudo-differential operators (see [9], 1.2.3.5) we conclude that the second integral in the left hand side of (2.2) is a bounded linear operator from $W^{p,2}(E_{|D})$ to $W^{p,2}(E_{|D})$.

Thus, to obtain (2.2) it suffices to pass to the limit in (2.3) for $N \to \infty$. □

Remark 2.5. The boundary integral in the left hand side of (2.2) does not depend on the choice of the Green operator G_P.

Corollary 2.6. *Let $\mathcal{L}$ be a bilateral fundamental solution of the operator P on X. Then the boundary integral in (2.2) is a (bounded) projection from $W^{m,2}(E_{|D})$ onto $S_P^{m,2}(D)$; and for every $f \in W^{m-p,2}(F_{|D})$ $(m \geq p)$ the integral $\int_D < \mathcal{L}(x,y), f(y) >_y dy$ is a $W^{m,2}(E_{|D})$-solution of equation (1.1) in D.*

Proof. Since the derivatives $D^\alpha u$ $(|\alpha| \leq p-1)$ have natural boundary values $D^\alpha u_{|\partial D} \in W^{m-|\alpha|-1/2,2}(E_{|\partial D})$, it is easy to see from [16] (Proposition 9.4) that the boundary integral in (2.2) does not depend on the choice of the Green operator G_P. Therefore, choosing as G_P the Green operator provided by Lemma 1.3, and using boundedness theorem for potential (co-boundary) operators on a manifold with boundary ([9], 2.3.2.5) one can conclude that the boundary integral in (2.2) defines a bounded linear operator from $W^{m,2}(E_{|D})$ to $W^{m,2}(E_{|D})$. Hence the statement follows from the properties of bilateral fundamental solutions of elliptic differential operators.

□

Remark 2.7. All the discussion above can be repeated, with small technical changes, under weaker smoothness assumptions on ∂D.

If the operator P is overdetermined, it may happen that there are no right (in particular bilateral) fundamental solutions.

Example 2.8. If P is the CAUCHY-RIEMANN system $\overline{\partial}$ in $\mathbb{C}^n$ ($\cong \mathbb{R}^{2n}$), $n > 1$, then there are no right fundamental solutions of P (due to the theorem on removability of compact singularities of holomorphic functions in $\mathbb{C}^n$ for $n > 1$). As a left fundamental solution of the CAUCHY - RIEMANN system we can take $\mathcal{L}(\zeta, z) = {}^tP^*(\zeta)\Phi(\zeta, z)$ where Φ is the standard fundamental solution of the Laplace operator in $\mathbb{R}^{2n}$ and $\zeta, z \in \mathbb{C}^n$. In this case (2.2) is the MARTINELLI - BOCHNER formula (see [1]) and the boundary integral in (2.2) is the MARTINELLI -BOCHNER integral. It is known that the MARTINELLI -BOCHNER integral gives harmonic but, in general, not holomorphic function everywhere outside of ∂D. Hence it is not a projection from $W^{m,2}(E_{|D})$ onto $S^{m,2}_{\overline{\partial}}(D)$. Moreover, the integral $u(x) = \int_D < \mathcal{L}(x, y), f(y) >_y dy$ is not solution of the equation $\overline{\partial}u = f$ in the domain D.

ROMANOV [11] proved that, if D is a bounded domain in $\mathbb{C}^n$, the limit $\lim_{\nu\to\infty} M^\nu$ of iterations of the MARTINELLI-BOCHNER integral M in the SOBOLEV space $W^{1,2}(D)$ exists; and that this limit is a projection from $W^{1,2}(D)$ onto the space of holomorphic $W^{1,2}(D)$ - functions (i.e. onto $S^{1,2}_{\overline{\partial}}(D)$). Using the iterations he also obtained a multi-dimensional analogue of the CAUCHY-GREEN formula in the plane, and, as a corollary, an explicit formula for solving the equation $\overline{\partial}u = f$ in pseudo-convex domains in $\mathbb{C}^n$.

It turns out that the convergence of the iterations is not a property which holds only for the MARTINELLI -BOCHNER integral. For example, in [14] the theorem on iterations was proved for special GREEN integrals of matrix factorizations of the Laplace operator in $\mathbb{R}^n$. In the next two sections we will prove a theorem on iterations for special GREEN integrals associated to general elliptic operators.

3. A theorem on iterations

Let $P \in do_p(E \to F)$ and let us denote by $\Delta \in do_{2p}(E \to E)$ the differential operator P^*P. The operator Δ is a determined elliptic operator of order $2p$ if and only if P is elliptic of order p. We assume that Δ is elliptic and has a bilateral fundamental solution Φ on X. As we noted before, this is always the case if we allow X to be taken sufficiently small or when we assume that the coefficients of P are real analytic. Then $\mathcal{L}(x, y) = {}^tP^*(y, D)\Phi(x, y)$ is a left fundamental solution of $P(x, D)$ on X.

Let D be an (open) relatively compact domain in X, with smooth boundary ∂D as in Section 2.. Having fixed a DIRICHLET system $\{B_j\}_{j=0}^{p-1}$ of order $(p-1)$ on ∂D as in Definition 2.2, we denote by G_P the corresponding GREEN operator given by Lemma 2.3. Then we define the operators M and T by setting, for $u \in W^{p,2}(E_{|D})$, $f \in$

$L^2(F_{|D})$,

$$(Mu)(x) = -\int_{\partial D} G_P({}^tP^*(y,D)\Phi(x,y), u(y)) \quad (x \in X\backslash\partial D),$$

$$(3.1) \qquad (Tf)(x) = \int_D < {}^tP^*(y,D)\Phi(x,y), f(y) >_y dy \quad (x \in X).$$

By Theorem 2.4, we have

$$(3.2) \qquad (Mu)(x) + (TPu)(x) = \begin{cases} u(x), & x \in D, \\ 0, & x \in X \setminus \overline{D} \end{cases}$$

for every $u \in W^{p,2}(E_{|D})$

Analogous to the Martinelli- Bochner integral, for every $u \in W^{p,2}(E_{|D})$ the integral Mu defines a $W^{p,2}(E_{|D})$-section which is only "harmonic", i.e. $\Delta Mu = 0$ everywhere outside of ∂D, while in general $PMu \neq 0$. By Corollary 2.6 we have

Proposition 3.1. *The integrals M and TP given above define linear bounded operators from $W^{m,2}(E_{|D})$ to $W^{m,2}(E_{|D})$ $(m \geq p)$.*

In particular, it is possible to consider iterations of the integrals M and TP in the Sobolev spaces $W^{m,2}(E_{|D})$ $(m \geq p)$

In order to prove his theorem on iterations for the Martinelli -Bochner integral A.V. Romanov constructed in [11] a suitable scalar product in the space $W^{1,2}(D)$. We follow his approach in our more general case.

Let us assume that we can construct in the Hilbert space $W^{m,2}(E_{|D})$ a scalar product $H_m^P(.,.)$ for which the following properties hold:

(I) *For every $u \in W^{m,2}(E_{|D})$: $H_m^P(Mu,u) \geq 0$, $H_m^P(TPu,u) \geq 0$.*

(II) *The topologies induced in $W^{m,2}(E_{|D})$ by $H_m^P(.,.)$ and by the standard scalar product of $W^{m,2}(E_{|D})$ are equivalent.*

In Section 4, by choosing special fundamental solutions, we will construct such a scalar product $H_p^P(.,.)$ in the Hilbert space $W^{p,2}(E_{|D})$.

In the remaining part of this section we will show that existence of a scalar product with properties (I) and (II) implies the convergence of iterations of the integrals M and TP (cf. [14]).

The kernels $\ker M$ and $\ker TP$ of the operators M and TP are closed subspaces of $W^{m,2}(E_{|D})$, therefore they are Hilbert spaces (with the Hermitian structure induced from $W^{m,2}(E_{|D})$). If S is a closed subspace of $W^{m,2}(E_{|D})$, we denote by $\Pi(S)$ the orthogonal projection with respect to $H_m^P(.,.)$ from $W^{m,2}(E_{|D})$ to S.

Theorem 3.2. *Assume that a scalar product $H_m^P(.,.)$ is defined in the space $W^{m,2}(E_{|D})$, for which* (I) *and* (II) *hold. Then*

$$\lim_{\nu\to\infty} M^\nu = \Pi(\ker TP), \quad \lim_{\nu\to\infty} (TP)^\nu = \Pi(\ker M)$$

in the strong operator topology in $W^{m,2}(E_{|D})$.

Proof. By (II) the space $W^{m,2}(E_{|D})$, with the scalar product $H_m^P(.,.)$, is a complex Hilbert space. Then (I) and (3.2) imply that the operators M and TP are self-adjoint in $W^{m,2}(E_{|D})$ with respect to the scalar product $H_m^P(.,.)$, and that $0 \le M \le Id, 0 \le TP \le Id$ (where Id stands for the identity operator on $W^{m,2}(E_{|D})$).

The spectral theorem for bounded self -adjoint operators yields

$$M^\nu = \int_0^1 \lambda^\nu dE_\lambda,\ (TP)^\nu = \int_0^1 (1-\lambda)^\nu dE_\lambda \tag{3.3}$$

where $\{E_\lambda\}_{0\le\lambda\le1}$ is a resolution of the identity in the Hilbert space $W^{m,2}(E_{|D})$ corresponding to the operator M and the scalar product $H_m^P(.,.)$.

Passing to the limit in (3.3) one obtains

$$\lim_{\nu\to\infty} M^\nu = \widetilde{E}_1,\quad \lim_{\nu\to\infty}(TP)^\nu = \widetilde{E}_0$$

where $\widetilde{E}_0 = E_{+0} - E_{-0}, \widetilde{E}_1 = E_{1+0} - E_{1-0}$ are the orthogonal projections from $W^{m,2}(E_{|D})$ onto the eigenspaces $V(0)$,$V(1)$ corresponding to the eigenvalues 0 and 1 of the operator M. Finally, (3.2) implies that $V(0) = \ker M, V(1) = \ker TP$. □

Corollary 3.3. *Under the hypotheses of* Theorem 3.2*, for every* $u \in W^{m,2}(E_{|D})$ $(m \ge p)$ *the following formulae hold:*

$$u = \lim_{\nu\to\infty} M^\nu u + \sum_{\mu=0}^{\infty} M^\mu(TPu), \tag{3.4}$$

$$u = \lim_{\nu\to\infty} (TP)^\nu u + \sum_{\mu=0}^{\infty} (TP)^\mu(Mu) \tag{3.5}$$

where the limits and the series in the right hand sides converge in the $W^{m,2}(E_{|D})$*-norm.*

Proof. Formula (3.2) implies that for every $\nu \in \mathbb{N}$

$$u = M^\nu u + \sum_{\mu=0}^{\nu-1} M^\mu(TPu) = (TP)^\nu u + \sum_{\mu=0}^{\nu-1}(TP)^\mu(Mu). \tag{3.6}$$

Using Theorem 3.2 we can pass to the limit for $\nu \to \infty$ in (3.6), obtaining (3.4) and (3.5). □

4. Construction of the projection $\Pi(S_P^{p,2}(D))$

In this section we construct a scalar product $H_p^P(\cdot,\cdot)$ on $W^{p,2}(E_{|D})$ satisfying (I), (II) of Section 3. This will be obtained by the use of a fundamental solution of $\Delta = P^*P$ enjoying special properties at the boundary of a subdomain Y of X.

Throughout this section we will assume that D is a relatively compact connected open subset of X, with a smooth boundary ∂D of class C^∞.

Since the Green integrals do not depend on the choice of the Dirichlet system $\{B_j\}$ on ∂D, in this section we can as well set $B_j = I_k \frac{\partial^j}{\partial n^j}$.

Proposition 4.1. *Assume that the operator $\Delta \in do_{2p}(E \to E)$ admits a bilateral fundamental solution Φ on X. Then for every domain $Y \Subset X$, with $\partial Y \in C^\infty$, there exists a unique bilateral fundamental solution $\Phi_Y(x, y)$ of the operator Δ in Y such that*

(1) *Φ_Y extends to a smooth function on $(\overline{Y} \times Y) \backslash \{(x, x) | x \in Y\}$;*

(2) *$(\frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi_Y(x, y))_{|x \in \partial Y} = 0$ for every $y \in Y$, every multi-index α, and $0 \le j \le p - 1$.*

Moreover, the function $\gamma = \Phi - \Phi_Y$ extends to a smooth function on $(\overline{Y} \times Y) \cup (Y \times \overline{Y})$.

Proof. The proof of Proposition 4.1 relies on the fact that the existence of a bilateral fundamental solution Φ of Δ in X implies existence and uniqueness of the Dirichlet problem for Δ on every subdomain D of X:

Lemma 4.2. *For every $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p - 1)$ there exists a (unique) section $\psi \in S^{m,2}_\Delta(D)$ such that $(B_j \psi)_{|\partial D} = \psi_j$ $(0 \le j \le p - 1)$.*

Proof. Let $\psi \in S^{m,2}_\Delta(D)$ be such that $B_j \psi = 0$ on ∂D $(0 \le j \le p - 1)$. Then there is a sequence $\{\psi_\nu\}$ of smooth functions with compact support in D such that $\lim_{\nu\to\infty} \psi_\nu = \psi$ in the $W^{p,2}(E_{|D})$-norm. Now using Stokes' formula, one has:

$$0 = \int_D (\psi, \Delta\psi)_x dx = \lim_{\nu\to\infty} \int_D (\psi_\nu, \Delta\psi)_x dx =$$

$$= \lim_{\nu\to\infty} \int_D (P\psi_\nu, P\psi)_x dx = \int_D (P\psi, P\psi)_x dx.$$

Hence $\psi \in S^{m,2}_P(D)$. By Theorem 2.4 we obtain that $\psi = M\psi = 0$ in the domain D. This proves the uniqueness of the Dirichlet problem.

We denote by $W^{p,2}_o(E_{|D})$ the space

$$W^{p,2}_o(E_{|D}) = \{u \in W^{p,2}(E_{|D}) : B_j u = 0 \ on \ \partial D \ for \ 0 \le j \le p - 1\}.$$

Because Δ is elliptic, we have the classical Gårding inequality:

$$\|u\|^2_{W^{p,2}(E_{|D})} \le c_0 \int_D (Pu, Pu)_x dx + \lambda_0 \|u\|^2_{L^2(E_{|D})} \quad (u \in W^{p,2}_o(E_{|D}))$$

for constants $c_0, \lambda_0 > 0$ which do not depend on u.

As we noted before, Theorem 2.4 implies that $u = 0$ if $u \in W^{p,2}_o(E_{|D})$ and $Pu = 0$ in D. Let us prove now that we can find a constant $c > 0$ such that for every

$u \in W_o^{p,2}(E_{|D})$ we have

$$\|u\|^2_{W^{p,2}(E_{|D})} \leq c \int_D (Pu, Pu)_x dx.$$

We argue by contradiction. If there is no such a constant then we can find a sequence $\{u_\nu\} \subset W_o^{p,2}(E_{|D})$ such that

$$\|u_\nu\|_{W^{p,2}(E_{|D})} = 1, \quad \|Pu_\nu\|_{L^2(F_{|D})} < 2^{-\nu}.$$

Because the unit ball in a separable HILBERT space is weakly compact, we can assume that the sequence $\{u_\nu\}$ weakly converges to a section $u_\infty \in W_o^{p,2}(E_{|D})$. Clearly we have $Pu_\infty = 0$ in D and hence $u_\infty = 0$ by the discussion above. But the Gårding inequality yields

$$1 \leq 2^{-\nu} + \lambda_0 \|u_\nu\|_{L^2(E_{|D})} \text{ for every } \nu$$

and hence, because the inclusion $W_o^{p,2}(E_{|D}) \to L^2(E_{|D})$ is compact, and thus u_ν strongly converges to u_∞ in $L^2(E_{|D})$, we obtain

$$\|u_\infty\|_{L^2(E_{|D})} \geq \lambda_0^{-1}$$

contradicting $u_\infty = 0$.

Thus we proved that the HERMITIAN form

$$\int_D (Pu, Pv)_x dx$$

defines in the HILBERT space $W_o^{p,2}(E_{|D})$ a scalar product which is equivalent to the original one. Therefore for every $\varphi \in W^{-p,2}(E_{|D})$ there is a unique solution of

$$\text{(4.1)} \qquad \begin{cases} u \in W_o^{p,2}(E_{|D}) \\ \int_D (Pu, Pv)_x dx = \varphi(\overline{v}) \text{ for every } v \in W_o^{p,2}(E_{|D}) \end{cases}$$

Moreover, by the regularity theorem for elliptic systems, if $\varphi \in W^{m,2}(E_{|D})$, the solution u of (4.1) belongs to $W_o^{p,2}(E_{|D}) \cap W^{2p+m,2}(E_{|D})$.

Given $w \in W^{m,2}(E_{|D})$, with $m \geq p$, the map

$$\mathcal{D}(E_{|D}) \ni v \to \int_D (w, \Delta v)_x dx$$

extends to a continuous anti-$\mathbb{C}$-linear functional on $W_o^{p,2}(E_{|D})$ and defines an element $\varphi \in W^{m-2p,2}(E_{|D})$. If u is a solution of (4.1) for w, then $\psi = w - u \in W^{m,2}(E_{|D})$, $\Delta\psi = 0$ in D, and $B_j\psi = B_j w$ on ∂D.

The proof of Lemma 4.2 is complete. □

Using the lemma, we obtain the fundamental solution Φ_Y in Y by subtracting from Φ the solution γ of the DIRICHLET problem

$$\begin{cases} \Delta(x)\gamma(x,y) = 0, \; x \in Y, y \in Y, \\ \frac{\partial^j}{\partial n_x^j}\gamma(x,y) = \frac{\partial^j}{\partial n_x^j}\Phi(x,y), \; x \in \partial Y, y \in Y, \quad (0 \leq j \leq p-1). \end{cases}$$

The solution smoothly depends on $y \in Y$ and one easily checks that $\Phi_Y = \Phi - \gamma$ satisfies the conditions set forth in the statement.

We turn now to the proof of the regularity of γ.

The fact that $\gamma \in C^\infty(\overline{Y} \times Y)$ follows from the regularity up to the boundary of the solution of a DIRICHLET problem with smooth data. The regularity of γ in $Y \times \overline{Y}$ is a consequence of the interior regularity of solutions of elliptic systems and the existence and uniqueness results for the DIRICHLET problem in SOBOLEV spaces of negative order (cf. [8], ch. 2, §6).

Let ρ be a defining function for Y. For every nonnegative integer r, define the spaces

$$\Xi^r(E_{|Y}) = \{u \in L^2(E_{|Y}) : \rho^{|\alpha|} D^\alpha u \in L^2(E_{|Y}) \ for \ |\alpha| \le r\}.$$

They are HILBERT spaces with the norm

$$\|u\|_{\Xi^r(E_{|Y})} = \sum_{|\alpha| \le r} \|\rho^{|\alpha|} D^\alpha u\|_{L^2(E_{|Y})}.$$

Then $\Xi^{-r}(E_{|Y})$ is defined as the strong dual of $\Xi^r(E_{|Y})$: it can be identified to a subspace of $D'(E_{|Y})$ because $D(E_{|Y})$ is dense in $\Xi^r(E_{|Y})$ for every integer $r \ge 0$. The definition of $\Xi^r(E_{|Y})$ for general $r \in \mathbb{R}$ is obtained by interpolation.

Next we introduce the HILBERT spaces

$$D_\Delta^{-r}(Y) = \{u \in W^{-r,2}(E_{|Y}) : \Delta u \in \Xi^{-r-2p}(E_{|Y})\},$$

endowed with the graph norm, for $r \ge 0$.

By the trace theorem (Theorem 6.5, p. 187 in [8]) the map

$$C^\infty(E_{|\overline{Y}}) \ni u \to \oplus_{j=0}^{p-1}(B_j u) \in \oplus_{j=0}^{p-1}(C^\infty(E_{|\partial Y}))$$

uniquely extends to a continuous linear map

$$D_\Delta^{-r}(Y) \ni u \to \oplus_{j=0}^{p-1}(B_j u) \in \oplus_{j=0}^{p-1}(W^{-r-j-1/2,2}(E_{|\partial Y}))$$

when $r + 1/2 \notin \mathbb{Z}$ and in this case the DIRICHLET problem

$$\begin{cases} \Delta u = f \quad in \ Y, \\ \frac{\partial^j}{\partial n^j} u = \psi_j \ on \ \partial Y, \ for \ 0 \le j \le p-1, \\ u \in D_\Delta^{-r}(Y) \end{cases}$$

has a unique solution for $f \in \Xi^{-r-2p}(E_{|Y})$ and $\psi_j \in W^{-r-j-1/2,2}(E_{|\partial Y}))$ (this is Theorem 6.6, p. 190 in [8]).

To apply the general result to our special situation, we note that for every fixed $\varepsilon > 0$, and every multi-index α

$$Y \ni x \to D_y^\alpha \Phi(x,y)$$

defines an element of $W^{2p-n/2-|\alpha|-\varepsilon,2}(E_{|Y})$ and $\Delta(x) = D_y^\alpha \delta(x-y) \otimes Id_E$ belongs to $\Xi^{n/2-|\alpha|-\varepsilon}(E_{|Y})$, uniformly for $y \in \overline{Y}$.

Having fixed α, we choose $r_\alpha \geq 0$ with $r_\alpha < 2p - n/2 - |\alpha|$ and $r_\alpha + 1/2 \notin \mathbb{Z}$. Since $\{D_y^\alpha \Phi(x,y) | y \in \overline{Y}\}$ is bounded in $D_\Delta^{-r_\alpha}(Y)$, also $\{\frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi(x,y) | y \in \overline{Y}\}$ is bounded in $\oplus W^{-r_\alpha - j - 1/2, 2}(E_{|\partial Y})$.

If $\widetilde{\gamma_\alpha}$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta(x)\widetilde{\gamma_\alpha}(x,y) = 0, \ x \in Y, y \in Y, \\ \frac{\partial^j}{\partial n_x^j}\widetilde{\gamma_\alpha}(x,y) = \frac{\partial^j}{\partial n_x^j} D_y^\alpha \Phi(x,y), \ x \in \partial Y, y \in Y, \quad (0 \leq j \leq p-1). \\ \widetilde{\gamma_\alpha}(.,y) \in D_\Delta^{-r_\alpha}(Y), \end{cases}$$

then $D_x^\beta \widetilde{\gamma_\alpha}$ is a bounded function of $y \in \overline{Y}$ for every multi-index β while x belongs to a compact subset of Y. Since $\widetilde{\gamma_\alpha} = D_y^\alpha \gamma(x,y)$ for $y \in Y$, the last part of the statement follows. □

Remark 4.3. In fact, one could prove more precise regularity of γ outside of diagonal of $\partial Y \times \partial Y$, together with bounds for the growth of its derivatives when (x,y) approaches the singularities (cf. [13], ch.VI, §4). However, the results obtained above suffice for our purposes.

We fix a domain Y with a C^∞-smooth boundary ∂Y such that $D \Subset Y \Subset X$. Let $\widetilde{S}_\Delta^{m,2}(Y \backslash \overline{D})$ $(m \geq p)$ be the Hilbert space of functions $v \in S_\Delta^{m,2}(Y \backslash \overline{D})$ such that $\frac{\partial^j v}{\partial n^j} = 0$ on ∂Y $(0 \leq j \leq p-1)$. We obtain a linear isomorphism

$$\widetilde{S}_\Delta^{m,2}(Y \backslash \overline{D}) \ni v \quad \overset{\mathcal{R}^+}{\longrightarrow} \quad \oplus_{j=0}^{p-1}(B_j v)_{|\partial D} \in \quad \oplus_{j=0}^{p-1}(W^{m-j-1/2,2}(E_{|\partial D})).$$

Composing $(\mathcal{R}^+)^{-1}$ with the trace operator

$$W^{m,2}(E_{|D}) \ni u \quad \overset{\mathcal{R}^-}{\longrightarrow} \quad \oplus_{j=0}^{p-1}(B_j u)_{|\partial D} \in \quad \oplus_{j=0}^{p-1}(W^{m-j-1/2,2}(E_{|\partial D})).$$

we obtain a continuous linear map

$$W^{m,2}(E_{|D}) \ni u \to S(u) \in \widetilde{S}_\Delta^{m,2}(Y \backslash \overline{D}).$$

For $u \in W^{p,2}(E_{|D})$, $f \in L^2(F_{|D})$, and $g \in L^2(F_{|Y \backslash D})$ we introduce now the following notations:

$$M_Y u(x) = -\sum_{j=0}^{p-1} \int_{\partial D} < (C_j {}^t P^*)(y) \Phi_Y(x,y), B_j u >_y ds \ (x \in Y \backslash \partial D),$$

$$M_Y S(u)(x) = -\sum_{j=0}^{p-1} \int_{\partial D} < (C_j {}^t P^*)(y) \Phi_Y(x,y), B_j S(u) >_y ds \ (x \in Y \backslash \partial D),$$

$$T_Y f(x) = \int_D < {}^t P^*(y) \Phi_Y(x,y), f(y) >_y dy \ (x \in Y),$$

$$T_Y g(x) = \int_{Y \backslash D} < {}^t P^*(y) \Phi_Y(x,y), g(y) >_y dy \ (x \in Y).$$

Because $(B_j S(u))_{|\partial D} = (B_j u)_{|\partial D}$ $(0 \le j \le p-1)$, we have $M_Y u = M_Y S(u)$.

In order to prove the Theorem on the limit of iterations of the integrals M_Y and $T_Y P$, we consider, for $u, v \in W^{m,2}(E_{|D})(m \ge p)$, the HERMITIAN form

$$H_p^P(u,v) = \int_D (Pu, Pv)_x dx + \int_{Y\setminus D} (PS(u), PS(v))_x dx.$$

Proposition 4.4. *The* HERMITIAN *form* $H_p^P(\cdot,\cdot)$ *defines a scalar product in* $W^{m,2}(E_{|D})$.

Proof. The coefficients of P are $C^\infty(\overline{Y})$ - functions, therefore, $PS(u) \in W^{m-p,2}$ $(E_{|Y\setminus D})$. Then, since $(\cdot,\cdot)_x$ is a HERMITIAN metric, to prove the statement it is sufficient to prove that $H_p^P(u,u) = 0$ implies $u \equiv 0$ in D.

Let $H_p^P(u,u) = 0$ then $u \in S_P^{m,2}(D)$, $S(u) \in S_P^{m,2}(Y\setminus\overline{D})$. Moreover, by definition $(B_j u)_{|\partial D} = (B_j S(u))_{|\partial D}$ $(0 \le j \le p-1)$. Then Theorem 3.2 of [17] implies that there exists a section $\mathfrak{U} \in S_P(Y)$ such that $\mathfrak{U}_{|D} = u, \mathfrak{U}_{|Y\setminus\overline{D}} = S(u)$. Then $\mathfrak{U} \in S_P^{m,2}(Y)$ and $\frac{\partial^j \mathfrak{U}}{\partial n^j} = 0$ for $0 \le j \le p-1$ on ∂Y. Therefore $\mathfrak{U} \equiv 0$ in Y (by the representation formula proved in Theorem 2.4), and in particular $u \equiv 0$ in D. □

Let $\{\widetilde{B_j}\}_{j=0}^{2p-1}$ be a DIRICHLET system of order $(2p-1)$ on ∂D (as above $b_j = j$), and let $\{\widetilde{C_j}\}_{j=0}^{2p-1}$ be the DIRICHLET system corresponding to $\{\widetilde{B_j}\}_{j=0}^{2p-1}$ with respect to the operator Δ and the GREEN operator as in Lemma 2.3. For $\psi_j \in W^{m+j-2p+1/2,2}(F_{j\,|\partial D})$ $(0 \le j \le 2p-1, m \ge 0)$ denote by $\mathcal{G}(\oplus\psi_j)$ the following integral:

$$\mathcal{G}(\oplus\psi_j)(x) = \sum_{j=0}^{2p-1} \int_{\partial D} < \widetilde{C_j}(y)\Phi(x,y), \psi_j >_y ds \ (x \in X\setminus\partial D).$$

Let $\mathcal{G}(\oplus\psi_j)^- = \mathcal{G}(\oplus\psi_j)_{|D}$, $\mathcal{G}(\oplus\psi_j)^+ = \mathcal{G}(\oplus\psi_j)_{|X\setminus\overline{D}}$. Then (cf. Lemma 2.7 in [15]) $\mathcal{G}(\oplus\psi_j)^- \in W^{m,2}(E_{|D})$, $\mathcal{G}(\oplus\psi_j)^+ \in W^{m,2}(Y\setminus D)$ and we have the jump formula

$$(\widetilde{B_j}\mathcal{G}(\oplus\psi_j)^-)_{|\partial D} - (\widetilde{B}_j\mathcal{G}(\oplus\psi_j)^+)_{|\partial D} = \psi_{2p-j-1}. \tag{4.2}$$

Lemma 4.5. *Let* $(Tf)^- = (Tf)_{|D}$, $(Tf)^+ = (Tf)_{|X\setminus\overline{D}}$. *Then for every* $f \in W^{p,2}(F_{|D})$ *we have*

$$(B_j(Tf)^-)_{|\partial D} - (B_j(Tf)^+)_{|\partial D} = 0,$$
$$({}^tC_j^* P(Tf)^-)_{|\partial D} - ({}^tC_j^* P(Tf)^+)_{|\partial D} = ({}^tC_j^* f)_{|\partial D}.$$

Proof. Using Stokes' formula we obtain for $x \notin \partial D$ and $f \in W^{p,2}(E_{|D})$:

$$Tf(x) = \int_D < \Phi(x,y), P^* f(y) >_y dy - \int_{\partial D} \sum_{j=0}^{p-1} < {}^tB_j^*(y)\Phi(x,y), {}^tC_j{}^*(y) f(y)). \tag{4.3}$$

Because $P^* f \in L^2(E_{|D})$, the first integral in the right hand side defines a section in $W^{2p,2}(E_{|Y})$. Indeed the fundamental solution Φ is a pseudo-differential operator of order $(-2p)$ on X. Thus it does not contribute to the jumps of the derivatives of Tf on ∂D up to order $(2p-1)$. The statement of the lemma is then a consequence of the jump formula (4.2), after nothing that $\{-C_j{}^tP^*, {}^tB_j^*\}_{j=0}^{p-1}$ is the DIRICHLET system corresponding to the DIRICHLET system $\{B_j, {}^tC_j^*P\}_{j=0}^{p-1}$ with respect to Δ in Lemma 2.3 (cf. [15], Theorem 4.4). □

Remark 4.6. In particular, if $f \in W^{p,2}(F_{|D})$ has compact support in D, then $Tf \in W^{2p,2}(E_{|Y})$.

Let $(T_Y g)^+ = (T_Y g)_{|Y\setminus\overline{D}}$, $(T_Y g)^- = (T_Y g)_{|D}$, and introduce similar notations for $T_Y f$ $(f \in L^2(F_{|D}),\ g \in L^2(F_{|Y\setminus D}))$.

Lemma 4.7. *For every $r \geq 0$ there exist a positive number $c(r)$ such that for every $f \in W^{r,2}(F_{|D})$ and $g \in W^{r,2}(F_{|Y\setminus D})$*

$$\|(T_Y f)^-\|^2_{W^{p+r,2}(E_{|D})} \leq c(r)\|f\|^2_{W^{r,2}(F_{|D})},$$

$$\|(T_Y f)^+\|^2_{W^{p+r,2}(E_{|Y\setminus D})} \leq c(r)\|f\|^2_{W^{r,2}(F_{|D})},$$

$$\|(T_Y g)^-\|^2_{W^{p+r,2}(E_{|D})} \leq c(r)\|g\|^2_{W^{r,2}(F_{|Y\setminus D})}.$$

Proof. By Proposition 4.1, $\gamma = \Phi - \Phi_Y$ is smooth in $(\overline{Y} \times Y) \cup (Y \times \overline{Y})$. Then

$$L^2(F_{|D}) \ni f \to \int_D < {}^tP^*(y)\gamma(x,y), f(y) >_y dy \in C^\infty(E_{|\overline{Y}})$$

and

$$L^2(F_{|Y\setminus\overline{D}}) \ni g \to \int_{Y\setminus D} < {}^tP^*(y)\gamma(x,y), g(y) >_y dy \in C^\infty(E_{|\overline{D}})$$

are linear and continuous maps. Therefore the proof of the estimates is reduced to the proof of the analogous estimates for T substituting T_Y.

When $0 \leq r < 1/2$, the estimates hold true because ${}^tP^*\Phi(x,y)$ is a pseudo-differential operator of order $(-p)$ on X and for general $r > 0$ by nothing that it has moreover the transmission property relative to every relatively compact open subset of X with a smooth boundary (cf. [9], 2.2.2 and 2.3.2). □

Remark 4.8. The lemma, together with the preceding remark, implies that $T_Y f \in W^{p,2}(E_{|Y})$ for every $f \in L^2(F_{|D})$. Indeed we can approximate $f \in L^2(F_{|D})$ by smooth sections with compact support in D in the L^2-norm. By the jump Lemma 4.5, $(T_Y f)^-$ and $(T_Y f)^+$ agree with their derivatives up to order $(p-1)$ on ∂D when f is smooth with compact support in D and hence by continuity the same is true when $f \in L^2(F_{|D})$.

Proposition 4.9. *For every $u, v \in W^{p,2}(E_{|D})$, $f \in L^2(F_{|D})$*

$$H_p^P(T_Y f, v) = \int_D (f, Pv)_x dx,$$

$$H_p^P(M_Y u, v) = \int_{Y\setminus D} (PS(u), PS(v))_x dx.$$

Proof. By integration by parts we obtain (cf. Lemma 2.3)

$$\int_D (f, Pv)_x dx - \int_D (P^* f, v)_x dx =$$

$$(4.4)\quad = \sum_{j=0}^{p-1} \int_{\partial D} < *_{F_j} B_j v, {}^t C_j^* f >_x \, ds \text{ for every } f \in W^{p,2}(F_{|D}),\ v \in W^{p,2}(E_{|D})$$

and analogously

$$(4.5)\quad \int_{Y\setminus D} (PS(u), PS(v))_x dx = -\sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) S(v), {}^t C_j^* PS(u) >_y \, ds =$$

$$= -\sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) v, {}^t C_j^* PS(u) >_y \, ds \text{ for every } u, v \in W^{p,2}(E_{|D}).$$

Let $u \in W^{2p,2}(E_{|D})$, $v \in W^{p,2}(E_{|D})$, and apply formula (4.4) for $f = Pu$. Then we obtain, using (4.4) and (4.5)

$$H_p^P(u, v) = \sum_{j=0}^{p-1} \int_{\partial D} < (*_{F_j} B_j) v, {}^t C_j^* Pu - {}^t C_j^* PS(u) >_y \, ds + \int_D (P^* Pu, v)_y dy.$$

Let $f \in \mathcal{D}(F_{|D})$. Then we can substitute $T_Y f$ for u in the formula above, to obtain

$$H_p^P(T_Y f, v) = \int_D (P^* P T_Y f, v)_y dy +$$

$$+ \sum_{j=0}^{p-1} \int_{\partial D} < *_{F_j} B_j v, {}^t C_j^* P(T_Y f)^- - {}^t C_j{}^* P(T_Y f)^+ >_y \, ds.$$

By Remark 4.6, $T_Y f \in W^{2p,2}(E_{|D})$ and thus the second summand in the right hand side of the last equality equals zero. Because

$$P^* P T_Y f(x) = P^* f(x) \ (x \in D),$$

we get

$$H_p^P(T_Y f, v) = \int_D (P^* f, v)_y dy = \int_D (f, Pv)_y dy.$$

Since $\mathcal{D}(F_{|D})$ is dense in $L^2(F_{|D})$, this formula holds for every $v \in W^{p,2}(E_{|D})$ and every $f \in L^2(F_{|D})$. Finally, (3.2) implies that $H_p^P(M_Y u, v) = H_p^P(u - T_Y Pu, v) = \int_{Y\setminus D} (PS(u), PS(v))_y dy$. □

Lemma 4.10. *For every* $u \in W^{m,2}(E_{|D})$ $(m \geq p)$

$$(T_Y Pu)(x) + (T_Y PS(u))(x) = \begin{cases} u(x), x \in D, \\ S(u)(x), x \in Y\setminus\overline{D}. \end{cases}$$

Proof. Since $\overline{Y} \subset X$, Theorem 2.4 implies that

$$-\int_{\partial(Y\setminus D)} G_P({}^tP^*(y)\Phi(x,y), S(u)(y)) + \int_{Y\setminus D} < {}^tP^*(y)\Phi(x,y), PS(u)(y) >_y dy =$$

$$= \begin{cases} S(u)(x), x \in Y\setminus D, \\ 0, x \in X\setminus(\overline{Y\setminus D}). \end{cases}$$

On the other hand, if $\gamma = \Phi - \Phi_Y$ then for every fixed point $x \in Y$ the integrals

$$\int_{\partial(Y\setminus D)} G_P({}^tP^*(y)\gamma(x,y), S(u)(y)) \ \ and \ \ \int_{Y\setminus D} < {}^tP^*(y)\gamma(x,y), PS(u)(y) >_y dy$$

are well defined. Then, since ${}^t\Delta(y)\gamma(x,y) = 0$ for $(x,y) \in Y \times Y$, Stokes' formula yields for $x \in Y$

$$-\int_{\partial(Y\setminus D)} G_P({}^tP^*(y)\gamma(x,y), S(u)(y)) + \int_{Y\setminus D} < {}^tP^*(y)\gamma(x,y), PS(u)(y) >_y dy = 0$$

Therefore, since $\frac{\partial^j S(f)}{\partial n^j} = 0$ on ∂Y

$$(T_Y PS(u))(x) - (M_Y S(u))(x) = \begin{cases} 0, & x \in D, \\ S(u)(x), & x \in Y\setminus\overline{D}. \end{cases} \tag{4.6}$$

Finally, $(B_j u)_{|\partial D} = (B_j S(u))_{|\partial D}$ by definition, hence $M_Y u = M_Y S(u)$. Now adding (3.2) and (4.6) we obtain the statement. □

Lemma 4.11. *The* Hilbert *spaces* $\widetilde{S}^{m,2}_{\Delta}(Y\setminus\overline{D})$, $\oplus_{j=0}^{p-1} W^{m-j-1/2,2}(F_{j\,|\partial D})$ *and* $S^{m,2}_{\Delta}(D)$ *are topologically isomorphic.*

Proof. Lemma 4.2 implies that for every $\oplus u_j \in \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$ there exist (unique) solutions $u \in S^{m,2}_{\Delta}(D)$ and $S(u) \in \widetilde{S}^{m,2}_{\Delta}(Y\setminus\overline{D})$ of the interior and exterior Dirichlet problems. Therefore, in order to prove the statement of the lemma it is sufficient to prove existence of constants $c_i > 0$ $(1 \le i \le 4)$ such that for every $\oplus u_j \in \oplus W^{m-j-1/2,2}(F_{j\,\partial D})$

$$c_1 \|u\|^2_{W^{m,2}(E_{|D})} \le \sum_{j=0}^{p-1} \|u_j\|^2_{W^{m-j-1/2,2}(F_{j\,|\partial D})} \le c_2 \|u\|^2_{W^{m,2}(E_{|D})}$$

$$c_3 \|S(u)\|^2_{W^{m,2}(E_{|Y\setminus D})} \le \sum_{j=0}^{p-1} \|u_j\|^2_{W^{m-j-1/2,2}(F_{j\,|\partial D})} \le c_4 \|S(u)\|^2_{W^{m,2}(E_{|Y\setminus D})}. \tag{4.7}$$

The existence of the constants c_2, c_4 follows from the continuity of the restriction maps

$$\mathcal{R}^- : S^{m,2}_{\Delta}(D) \to \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$$

$$\mathcal{R}^+ : \widetilde{S}^{m,2}_{\Delta}(Y\setminus\overline{D}) \to \oplus W^{m-j-1/2,2}(F_{j\,|\partial D})$$

where $\mathcal{R}^- u = \oplus(B_j u)_{|\partial D}, \mathcal{R}^+ S(u) = \oplus(B_j S(u))_{|\partial D}$. Since $\mathcal{R}^-, \mathcal{R}^+$ are one-to-one and onto (see Lemma 4.2), the existence of constants c_1, c_3 follows from the open mapping theorem. □

Proposition 4.12. *The topologies induced in $W^{p,2}(E_{|D})$ by $H_p^P(\cdot,\cdot)$ and by the standard scalar product are equivalent.*

Proof. Since the coefficients of P are $C^\infty(\overline{Y})$- functions then there are constants $c_5, c_6 > 0$ such that for every $u \in W^{p,2}(E_{|D})$

$$(Pu, Pu)_x \le c_5 \sum_{|\alpha|\le p} (D^\alpha u, D^\alpha u)_x, \ (PS(u), PS(u))_x \le c_6 \sum_{|\alpha|\le p} (D^\alpha S(u), D^\alpha S(u))_x.$$

On the other hand, Lemma 4.11 (see (4.7)) implies that

$$\|S(u)\|^2_{W^{p,2}(E_{Y\setminus \overline{D}})} \le c_2 (c_3)^{-1} \|u\|^2_{W^{p,2}(E_{|D})}.$$

Hence

$$H_p^P(u,u) \le (c_5 + c_6 c_2 (c_3)^{-1}) \|u\|^2_{W^{p,2}(E_{|D})}.$$

Conversely, Lemmata 4.7 and 4.10 imply that

$$(1/2)\|u\|^2_{W^{p,2}(E_{|D})} \le \|T_Y P u\|^2_{W^{p,2}(E_{|D})} + \|T_Y PS(u)\|^2_{W^{p,2}(E_{|D})} \le$$

$$\le c(0)\|Pu\|^2_{L^2(F_{|D})} + c(0)\|PS(u)\|^2_{L^2(F_{|Y\setminus D})} = c(0) H_p^P(u,u),$$

which had to be proved. □

In the following theorem $\widetilde{S}_P^{p,2}(Y\setminus\overline{D})$ stands for the subspace of $W^{p,2}(E_{|D})$ which consists of functions $u \in W^{p,2}(E_{|D})$ such that $PS(u) = 0$ in $(Y\setminus\overline{D})$.

Theorem 4.13. *In the strong operator topology in $W^{p,2}(E_{|D})$*

$$\lim_{\nu\to\infty} M_Y^\nu = \Pi(S_P^{p,2}(D)),$$

$$\lim_{\nu\to\infty} (T_Y P)^\nu = \Pi(\widetilde{S}_P^{p,2}(Y\setminus\overline{D})).$$

Proof. First, Propositions 4.9 and 4.12 imply that (I) and (II) hold for $H_p^P(\cdot,\cdot)$ and M_Y, $T_Y P$. Second, Proposition 4.9 implies that $ker T_Y P = S_P^{p,2}(D)$. Third, Proposition 4.9, (3.2) and Lemma 4.10 imply that $M_Y u = 0$ if and only if $S(u) \in S_P^{p,2}(Y\setminus\overline{D})$. Hence the theorem follows from Theorem 3.2. □

Remark 4.14. Let the operator P satisfy the so-called Uniqueness Condition in the small on X, i.e. $Pu = 0$ in a domain $D \subset X$ and $u = 0$ in an open subset of D imply $u \equiv 0$ in D. Then, if ∂D is connected, the Uniqueness Theorem for the Cauchy problem for systems with injective symbols (see [15], Theorem 2.8), implies that $\widetilde{S}_P^{p,2}(Y\setminus\overline{D}) = W_0^{p,2}(E_{|D})$. For instance, the Uniqueness Condition holds if the coefficients of the operator P are real analytic.

5. Solvability conditions for the equation $Pu = f$

In this section we will use Theorem 4.13 to investigate solvability of equation (1.1). In particular, when (1.1) is solvable we will obtain an expression of the solution by means of a series that can be computed from the data.

Let $P \in do_p(E \to F)$ be an elliptic operator of order p, as in Section 2.. We assume that P is included into some elliptic complex of differential operators on X:

$$C^\infty(E) \xrightarrow{P} C^\infty(F) \xrightarrow{P^1} C^\infty(G) \tag{5.1}$$

for a trivial vector bundle $G = X \times \mathbb{C}^t$ and $P^1 \in do_{p_1}(F \to G)$. The assumptions mean that

$$P^1 \circ P = 0$$

and that

$$E_x \overset{\sigma_p(P)(x,\zeta)}{\longrightarrow} F_x \overset{\sigma_{p_1}(P^1)(x,\zeta)}{\longrightarrow} G_x$$

is an exact sequence for every $x \in X$ and $\zeta \in \mathbb{R}^n \setminus \{0\}$. According to [12] (cf. also [2]) this is possible under rather general assumptions on P.

Note that the condition $P^1 f = 0$ is necessary in order that (1.1) be solvable. We formulate now

Problem 5.1. Let $r \geq 0$, $0 \leq m \leq p + r$, and $f \in W^{r,2}(F_{|D})$ be a given section. It is required to find a section $u \in W^{m,2}(E_{|D})$ such that $Pu = f$ in D.

Let, as before, $\{B_j\}_{j=0}^{p-1}$ be a Dirichlet system of order $(p-1)$ on ∂D, $\{C_j\}_{j=0}^{p-1}$ be the Dirichlet system associated to $\{B_j\}_{j=0}^{p-1}$ as in Lemma 2.3, and let, for $r \geq 0$,

$$\mathfrak{H}^{r,2}(D) = \{g \in W^{r,2}(F_{|D}) \text{ such that } P^* g = 0\, P^1 g = 0 \text{ in } D, \text{ weakly satisfying}$$
$$\text{the boundary conditions } ({}^t C_j^* g)_{|\partial D} = 0,\ 0 \leq j \leq p-1\}.$$

We call the $\mathfrak{H}^{r,2}(D)$ *harmonic spaces* (for complex (5.1)). By the ellipticity assumptions, $\mathfrak{H}^{r,2}(D) \subset C^\infty(F_{|D})$. It is not difficult to show that for the Dolbeault complex this definition of the harmonic space $\mathfrak{H}^{0,2}(D)$ is equivalent to the one given in [5].

We denote by R_Y the series

$$R_Y = \sum_{\mu=0}^{\infty} M_Y^\mu T_Y.$$

For every $r \geq 0$ we set

$$domR_Y^{p,r} = \{g \in L^2(F_{|D}) :\ R_Y g \text{ converges in the } W^{p,2}(E_{|D})\text{-norm},$$
$$\text{and } P(R_Y g) \in W^{r,2}(F_{|D})\}.$$

Then R_Y defines a linear operator $R_Y^{p,r} : domR_Y^{p,r} \to W^{p,2}(E_{|D})$. This series will play an essential role in our investigation of equation (1.1).

Proposition 5.2. *Let $(S^{p,2}_P(D))^\perp$ be the orthogonal complement of $S^{p,2}_P(D)$ in $W^{p,2}(E_{|D})$ with respect to $H^P_p(\cdot,\cdot)$. Then $Im(R^{(p,0)}_Y) = (S^{p,2}_P(D))^\perp$.*

Proof. If $f \in dom R^{(p,0)}_Y$ then $R_Y f \in W^{p,2}(E_{|D})$, and, since M_Y is continuous (see Proposition 3.1),

$$(5.2) \qquad M_Y R_Y f = M_Y \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu} M^\mu_Y T_Y f = \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu} M^{\mu+1}_Y T_Y f = R_Y f - T_Y f.$$

Therefore

$$(5.3) \qquad M^\nu_Y R_Y f = R_Y f - \sum_{\mu=0}^{\nu-1} M^\mu_Y T_Y f.$$

Passing to the limit for $\nu \to \infty$ in (5.3) we obtain that $\lim_{\nu\to\infty} M^\nu_Y R_Y f = R_Y f - R_Y f = 0$, i.e. $\Pi(S^{p,2}_P(D))R_Y f = 0$ and therefore $R_Y f \in (S^{p,2}_P(D))^\perp$.

Conversely, if $u \in (S^{p,2}_P(D))^\perp$ then (3.4) and Theorem 4.13 imply that $u = R_Y P u$. By Proposition 3.1 and Corollary 3.3 we have $Pu \in dom R^{(p,0)}_Y$. Therefore we conclude that $(S^{p,2}_P(D))^\perp \subset Im(R^{(p,0)}_Y)$. □

In particular Proposition 5.2 implies that $Im(R^{(p,r)}_Y) \subset (S^{p,2}_P(D))^\perp$.

By formula (3.4) the series R_Y defines the left inverse of P on $(S^{p,2}_P(D))^\perp$. In the following proposition we find a condition for R_Y to be also a right inverse operator of P.

Proposition 5.3. $\ker R^{(p,r)}_Y = 0$ *if and only if* $PR^{(p,r)}_Y = Id_{|dom R^{(p,r)}_Y}$.

Proof. If $f \in dom R^{(p,r)}_Y$ then $R_Y f \in W^{p,2}(E_{|D})$ and $PR_Y f \in dom R^{(p,r)}_Y$ by (3.4). Because $R^{(p,r)}_Y$ is a left inverse of P on $(S^{p,2}_P(D))^\perp$ and, due to Proposition 5.2, $Im R^{(p,r)}_Y \subset (S^{p,2}_P(D))^\perp$, we obtain $R^{(p,r)}_Y P R^{(p,r)}_Y = R^{(p,r)}_Y$. From this identity we deduce that $PR^{(p,r)}_Y = Id_{|dom R^{(p,r)}_Y}$ if $R^{(p,r)}_Y$ is injective, while the converse statement is obvious. □

Proposition 5.4. $\ker R^{(p,r)}_Y = \ker T_Y \cap dom R^{(p,r)}_Y$ $(r \geq 0)$.

Proof. Clearly $\ker T_Y \subset \ker R^{(p,r)}_Y$. The opposite inclusion follows from (5.2). □

Let us denote by $\mathfrak{N}^{r,2}_m(D)$ the set of all $f \in W^{r,2}(F_{|D})$ for which Problem 5.1 is solvable:

$$\mathfrak{N}^{r,2}_m(D) = \{f \in W^{r,2}(F_{|D}) : \text{there exists a section } u \in W^{m,2}(E_{|D}) \text{ such that } Pu = f \text{ in } D\}.$$

We obtain:

Proposition 5.5. *We have*

(1) $\mathfrak{N}^{r,2}_m(D) \subset S^{r,2}_{P^1}(D)$ $(m \geq 0)$;

(2) $\int_D (g, f)_x dx = 0$ *for every* $f \in \mathfrak{N}^{r,2}_m(D)$ *and every* $g \in \mathfrak{H}^{r,2}(D)$ $(m \geq p)$;

(3) $\mathfrak{N}^{r,2}_m(D) \subset dom R^{p,r}_Y$ $(m \geq p)$;

(4) $\ker T_Y \cap \mathfrak{N}^{r,2}_m(D) = 0$ $(m \geq p)$.

Proof. (1) is trivial, because (5.1) is a complex. Corollary 3.3 and Theorem reft.3.13 imply that $Pu = PR_Y Pu$ for $u \in W^{p,2}(E_{|D})$, i.e, (3) holds. To prove (2), we fix $f \in \mathfrak{N}^{r,2}_m(D)$ and a section $u \in W^{m,2}(E_{|D})$ such that $Pu = f$ in D.

For $\varepsilon > 0$ we set $D_\varepsilon = \{x \in D : \quad dist(x, \partial D) > \varepsilon\}$. Since the differential complex (5.1) is elliptic, $\mathfrak{H}^{r,2}(D) \subset C^\infty(F_{|D})$. Hence, for every $g \in \mathfrak{H}^{r,2}(D)$, we have:

$$\int_D (g, f)_x dx = \int_D (g, Pu)_x dx = \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (g, Pu)_x dx =$$

$$= \lim_{\varepsilon \to 0} (\int_{D_\varepsilon} (P^* g, u)_x dx - \int_{\partial D_\varepsilon} G_{P^*}(*_E u, g)_x dx) =$$

$$= \lim_{\varepsilon \to 0} \sum_{j=0}^{p-1} \int_{\partial D_\varepsilon} < (*_{F_j} B_j u), {}^t C^*_j g >_y ds = 0.$$

Therefore (2) holds.

Finally, if $f \in \ker T_Y \cap \mathfrak{N}^{r,2}_m(D)$ then (due to Proposition 5.4) $f \in \ker R^{p,r}_Y \cap \mathfrak{N}^{r,2}_m(D)$. Therefore $0 = PR_Y f = f$. □

Theorem 5.6. *Let* $r \geq 0$, $m = p$ *and* $f \in W^{r,2}(F_{|D})$. *Then* Problem 5.1 *is solvable if and only if*

(1) $f \in S^{r,2}_{P^1}(D) \cap dom R^{p,r}_Y$;

(2) $\int_D (g, f)_x dx = 0$ *for every* $g \in \mathfrak{H}^{r,2}(D)$.

Proof. The necessity follows from Proposition 5.5. In order to prove the converse statement we will use the following lemma.

Lemma 5.7. $\mathfrak{H}^{r,2}(D) = \ker T_Y \cap S^{r,2}_{P^1}(D)$ $(r \geq 0)$.

Proof. Let $f \in \mathfrak{H}^{r,2}(D)$. Then $f \in C^\infty(F_{|D})$. But for every $f \in kerP^* \cap C^\infty(F_{|D}) \cap L^2(F_{|D})$ and $x \in Y \backslash \partial D$ we have:

$$T_Y f(x) = \int_D < {}^t P^*(y) \Phi_Y(x, y), f(y) >_y dy =$$

$$= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} < {}^t P^*(y) \Phi_Y(x, y), f(y) >_y dy =$$

$$= \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} \sum_{j=0}^{p-1} < {}^t B_j^*(y)\Phi_Y(x,y), {}^t C_j^* f(y) >_y ds. \tag{5.4}$$

Therefore, since the weak boundary values $({}^t C_j^* f)_{|\partial D}$ equal to zero $(0 \leq j \leq p-1)$, the last limit in (5.4) is equal to zero.

Let us prove now the opposite inclusion. Since Φ_Y is a bilateral fundamental solution of the operator P^*P in Y then $\widetilde{\Phi}_Y(x,y) = \Phi_Y(y,x)$ is a bilateral fundamental solution of the operator ${}^t(P^*P)$ on Y. In particular, for every $v \in \mathcal{D}(E^*_{|D})$ we have

$$v(y) = \int_D < \Phi_Y(x,y), {}^t(P^*P)(x)v(x) >_x dx.$$

For every given section $f \in L^2(F_{|D})$ we can find a sequence $\{f_N\} \subset C(F_{|\overline{D}})$ such that $\lim_{N\to\infty} f_N = f$ in the $L^2(F_{|D})$- norm. Assume moreover that $f \in \ker T_Y \cap W^{r,2}(F_{|D})$. Then, for every $v \in \mathcal{D}(E^*_{|D})$ we have

$$\int_D < {}^t P^*(y)v(y), f(y) >_y dy = \lim_{N\to\infty} \int_D < {}^t P^*(y)v(y), f_N(y) >_y dy =$$

$$= \lim_{N\to\infty} \int_{D_y} < {}^t P^*(y) \int_{D_x} < \Phi_Y(x,y), {}^t(P^*P)(x)v(x) >_x dx, f_N(y) >_y dy =$$

$$= \lim_{N\to\infty} \int_D < T_Y f_N(x), {}^t(P^*P)(x)v(x) >_x dx.$$

By Lemma 4.7, $T_Y : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ is continuous and therefore $\{Tf_N\}$ converges in the $W^{p,2}(E_{|D})$-norm to $T_Y f = 0$. This shows that

$$\int_D < {}^t P^*(y)v(y), f(y) >_y dy = 0 \text{ for every } v \in \mathcal{D}(F_{|D}).$$

Hence $P^*f = P^1 f = 0$ if $f \in \ker T_Y \cap S^{r,2}_{P1}(D)$. Note that regularity theorem for elliptic systems gives in particular $\ker T_Y \cap S^{r,2}_{P1}(D) \subset C^\infty(F_{|D})$.

To complete the proof, we only need to show that (in the weak sense) $({}^t C_j^* f)_{|\partial D} = 0$ on ∂D $(0 \leq j \leq p-1)$ for $f \in \ker T_Y \cap S^{r,2}_{P1}(D)$. To this aim, we prove that

$$\lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^t C_j^* f(y) >_y ds = 0$$

for every $v^{(j)} \in C^\infty_{comp}(F_j^*)$.

Let $v^{(j)} \in C^\infty_{comp}(F_j^*)$. Then we fix a domain Ω with $D \Subset \Omega \Subset Y$, and find a section $v \in \mathcal{D}(E^*_{|\Omega})$ such that ${}^t B_j^* v = v^{(j)}$ on ∂D, and ${}^t B_i^* v = 0$ on ∂D, if $i \neq j$ (see [17], Lemma 28.2). Again we use representation formula:

$$v(y) = \int_\Omega < \Phi_Y(x,y), {}^t(P^*P)(x)v(x) >_x dx.$$

Since $P^* f = 0$ and $f \in C^\infty(F_{|D})$, arguing as before we have

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon} < v^{(j)}, {}^t C_j^* f(y) >_y ds = \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\sum_{i=0}^{p-1} < {}^t B_i^* v, {}^t C_i^* f(y) >_y ds =$$

$$= \lim_{\varepsilon\to 0}\int_{D_\varepsilon} < {}^t P^* v, f >_y dy = \int_D < {}^t P^* v, f >_y dy =$$

$$= \lim_{N\to\infty}\int_\Omega < T_Y f_N(x), {}^t(P^* P)(x) v(x) >_x dx.$$

Lemma 4.7 implies that $\lim_{N\to\infty}(T_Y f_N)_{|\Omega}$ converges in $W^{p,2}(F_{|\Omega})$ to $(T_Y f)_{|\Omega}$. However, due to Proposition 4.1 and Remark 4.8, $T_Y f = 0$ in D implies $T_Y f = 0$ in Y. Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon} < v^{(j)}, {}^t C_j^* f(y) >_y ds = 0.$$

The proof of the lemma is complete. □

Now we turn to the proof of Theorem 5.6. Since, under the hypothesis of the theorem, $R_Y f \in W^{p,2}(F_{|D})$, by Proposition 5.2

$$R_Y f = \lim_{\nu\to\infty} M_Y^\nu R_Y f + R_Y P R_Y f = R_Y P R_Y f. \tag{5.5}$$

In particular, $(f - P R_Y f) \in \ker R_Y^{p,r} \cap W^{r,2}(F_{|D})$, and, due to Proposition 5.4, $(f - P R_Y f) \in \ker T_Y \cap S_{P^1}^{r,2}(D)$. On the other hand, using Lemma 5.7 and the hypothesis of the theorem, we conclude that

$$\int_D (f - P R_Y f, f - P R_Y f)_x dx = 0.$$

Therefore $f = P R_Y f$, i.e. Problem 5.1 is solvable. □

As one can see from the proof of Theorem 5.6, if the equation $Pu = f$ is solvable in $W^{p,2}(E_{|D})$ then we obtain a formula for a solution of the equation:

$$u = R_Y f = \sum_{\nu=0}^{\infty} M_Y^\nu T_Y f.$$

In the case where $P = \overline{\partial}$ and M_Y is the Martinelli- Bochner integral such a formula was obtained by Romanov [11].

We conjecture that when the Poincarè lemma (local solvability) is valid for an elliptic complex, a solution in $W^{p,2}(E_{|D})$ can be found for every datum f in $W^{p,2}(F_{|D})$ satisfying the integrability conditions. If this is the case, the formula above produces rather explicitly a way to obtain a solution by successive approximations.

Remark 5.8. Proposition 5.2 and Theorem 4.13 imply that the solution $u = R_Y f$ of Problem 5.1 belongs to $(S_P^{p,2}(D))^\perp$ where $(S_P^{p,2}(D))^\perp$ is the orthogonal (with respect to $H_p^P(.,.)$) complement of $S_P^{p,2}(D)$ in $W^{p,2}(E_{|D})$, and is the unique solution belonging to this subspace.

We note that the general term $M_Y^\mu T_Y f$ of the series $R_Y f$ is infinitesimal in $W^{p,2}(E_{|D})$ for every $f \in L^2(F_{|D})$. This is a consequence of the theorem on iterations.

Proposition 5.9. *For every $f \in L^2(F_{|D})$, $\lim_{\nu\to 0} M_Y^\nu T_Y f = 0$ in the $W^{p,2}(E_{|D})$ norm, i.e. $T_Y f \in (S_P^{p,2}(D))^\perp$.*

Proof. It follows from Proposition 4.9 that

$$H_p^P(T_Y f, v) = \int_D (f, Pv)_x dx = 0,$$

if $v \in S_P^{p,2}(D)$. □

We also have

Proposition 5.10. *Let $f \in L^2(F_{|D})$. Then a necessary and sufficient condition for the convergence of the series R_Y in $W^{p,2}(E_{|D})$ is the convergence of the series*

$$\sum_{\mu=0}^{\infty} \|M_Y^\mu T_Y f\|^2_{W^{p,2}(E_{|D})}. \tag{5.6}$$

Proof. Since the scalar product $H_p^P(.,.)$ is equivalent to the usual one in $W^{p,2}(E_{|D})$ the convergence of the series (5.6) is equivalent to that of the series

$$\sum_{\mu=0}^{\infty} H_p^P(M_Y^\mu T_Y f, M_Y^\mu T_Y f).$$

Then the statement follows because M_Y is non negative and self-adjoint with respect to the scalar product $H_p^P(\cdot,\cdot)$. □

6. On the Poincarè Lemma for elliptic differential complexes

We investigate now conditions for the vanishing of the cohomology groups

$$H(W^{r,2}(F_{|D})) = S_{P^1}^{r,2}(D)/\mathfrak{N}_p^{r,2}(D)$$

of the complex (5.1).

From Theorem 5.6 and Proposition 5.5 of the previous section we have

Corollary 6.1. $H(W^{r,2}(F_{|D})) = 0$ $(r \geq 0)$ *if and only if*

(1) $S_{P^1}^{r,2}(D) \subset dom R_Y^{p,r}$;

(2) $\mathfrak{H}^{r,2}(D) = 0$.

Let us clarify the conditions in Corollary 6.1.

Proposition 6.2. *The natural map* $i : \ker T_Y \cap S^{r,2}_{P1}(D) \to H(W^{r,2}(F_{|D}))$ *is injective.*

P r o o f. This follows from statement (4) of Proposition 5.5. □

Proposition 6.3. $S^{r,2}_{P1}(D) \subset dom R^{p,r}_Y$ *if and only if the natural map* $i : \mathfrak{H}^{r,2}(D) \to H(W^{r,2}(F_{|D}))$ *is bijective.*

P r o o f. Assume that $S^{r,2}_{P1}(D) \subset dom_r R_Y$. Then formula (5.5) and Proposition 5.4 imply that, for every $f \in S^{r,2}_{P1}(D)$, the section $(f - PR_Y f)$ belongs to $\ker T_Y \cap S^{r,2}_{P1}(D)$. Obviously, $(f - PR_Y f)$ belongs to the same cohomology class as f. By Lemma 5.7, $\mathfrak{H}^{r,2}(D) = \ker T_Y \cap S^{r,2}_{P1}(D)$. Then the map i is surjective and, due to Proposition 6.2, is also injective.

On the other hand, if the natural map $i : \mathfrak{H}^{r,2}(D) \to H(W^{r,2}(F_{|D}))$ is surjective then, again using Lemma 5.7, for every $f \in S^{r,2}_{P1}(D)$, there exist sections $\widetilde{f} \in \ker T_Y \cap S^{r,2}_{P1}(D)$ and $u \in W^{p,2}(E_{|D})$ such that $f = \widetilde{f} + Pu$. In particular, due to Proposition 5.4, we obtain that $R_Y f = R_Y(\widetilde{f} + Pu) = R_Y Pu$. Now using Corollary 3.3 we conclude that the series $R_Y Pu$ converges in the $W^{p,2}(E_{|D})$-norm. Hence $R_Y(\widetilde{f} + Pu)$ also converges in $W^{p,2}(E_{|D})$-norm. Therefore, since $PR_Y f = PR_Y Pu = Pu$ and $Pu = f - \widetilde{f} \in W^{r,2}(F_{|D})$, we obtain that $f \in dom R^{p,r}_Y$. □

The triviality of the cohomology group $H(W^{r,2}(F_{|D}))$, implies, in particular, that the range $Im(P^{p,r})$ of the map $P^{p,r} : W^{p,2}(E_{|D}) \to W^{r,2}(F_{|D})$ is closed in $W^{r,2}(F_{|D})$. In the following statement $\overline{Im(P^{p,r})}$ stands for the closure of the range $Im(P^{p,r})$ in $W^{r,2}(F_{|D})$.

Proposition 6.4. *The range* $Im(P^{p,r})$ *is closed if and only if* $\overline{Im(P^{p,r})} \subset dom R^{p,r}_Y$ $(r \geq 0)$.

P r o o f. Let $f \in dom R^{p,r}_Y$. Then $f - PR_Y f$ belongs to $\ker R_Y$ by (refeq.4.5). Since $\ker R_Y = \ker T_Y$ by Proposition 5.4, we obtain that $dom R^{p,r}_Y = \ker T_Y \oplus Im(P^{p,r})$. If we assume that $\overline{Im(P^{p,r})} \subset dom R^{p,r}_Y$ we obtain a sum decomposition

$$\overline{Im(P^{p,r})} = (\ker T_Y \cap \overline{Im(P^{p,r})}) \oplus Im(P^{p,r}). \tag{6.1}$$

On the other hand, if $f \in (\ker T_Y \cap \overline{Im(P^{p,r})}$ then there exists a sequence $\{u_N\} \subset W^{p,2}(E_{|D})$ such that $\lim_{N\to\infty} Pu_N = f$ in the $L^2(F_{|D})$-norm. Hence, due to Proposition 4.9,

$$\|f\|^2_{L^2(F_{|D})} = \lim_{N\to\infty} \int_D (f, Pu_N)_x dx = \lim_{N\to\infty} H^P_p(T_Y f, u_N) = 0.$$

Therefore

$$(\ker T_Y \cap \overline{Im(P^{p,r})}) = 0$$

and, by (6.1), the range $Im(P^{p,r})$ is closed.

Conversely, by (3) in Proposition 5.5 we have $\overline{Im(P^{p,r})} = \mathfrak{N}_p^{r,2}(D) \subset dom R_Y^{p,r}$ and therefore the conclusion is obviously necessary. □

Using the integrals T_Y and M_Y we obtain simpler conditions for the first cohomology group of the complex (5.1) to be trivial in the case $r = 0$ and $m = p$. This is the case where solutions can be obtained with maximal global regularity. This applies for instance to the de Rham complex, but does not to the Dolbeault complex (see Example 7.4).

To simplify notations we will write $Im(P)$ instead of $Im(P^{p,0})$.

Proposition 6.5. $H(L^2(F_{|D})) = 0$ *if and only if*

(1) *the range* $Im(P)$ *of the map* $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ *is closed in* $L^2(F_{|D})$*;*

(2) $\mathfrak{H}^{0,2}(D) = 0$.

Proof. Necessity. Let $H(L^2(F_{|D})) = 0$ then $S^{0,2}_{P^1}(D) = Im(P)$. Hence, since $S^{0,2}_{P^1}(D)$ is a closed subspace of $L^2(F_{|D})$, $Im(P)$ is closed. The necessity of condition (2) of the theorem follows from Proposition 5.5.

Sufficiency. Let the range $Im(P)$ of the map $P : W^{p,2}(E_{|D}) \to L^2(F_{|D})$ be closed in $L^2(F_{|D})$. Then the continuous map

$$P : (S^{p,2}_P(D))^\perp \to Im(P)$$

is one-to-one. Now, since $Im(P)$ and $(S^{p,2}_P(D))^\perp)$ are closed subspaces of $L^2(F_{|D})$ and $W^{p,2}(E_{|D})$ respectively, the open map theorem implies that there exists a positive constant c such that

$$\|v\|_{W^{p,2}(E_{|D})} \leq c\|Pv\|_{L^2(F_{|D})}$$

for every $v \in (S^{p,2}_P(D))^\perp)$.

Therefore the Hermitian form

$$\widetilde{H}^P_p(u,v) = \int_D (Pu, Pv)_x dx$$

is a scalar product on $(S^{p,2}_P(D))^\perp$; and the topology induced in $(S^{p,2}_P(D))^\perp$ by this scalar product is equivalent to the original one.

Let f be a section in $S^{0,2}_{P^1}(D)$. Then the integral

$$\int_D (f, Pu)_x dx \ (v \in (S^{p,2}_P(D))^\perp))$$

defines a continuous linear functional on $(S^{p,2}_P(D))^\emptyset)$. Now, using Riesz representation theorem, we conclude that there exists $u \in (S^{p,2}_P(D))^\perp$ such that

$$\int_D (f, Pv)_x dx = \int_D (Pu, Pv)_x dx \tag{6.2}$$

for every $v \in (S^{p,2}_P(D))^\perp$. But then (6.2) holds for every $v \in W^{p,2}(E_{|D})$.

Furthermore, since for every $w \in C_0^\infty(F^*_{|D})$ the section $(*_F^{-1} w)$ belongs to $W^{p,2}(E_{|D})$, we have

$$\int_D < {}^tP^* w, f >_x dx = \int_D (f, P(*_F^{-1} w))_x dx = 0,$$

i.e. $P^*(f - Pu) = 0$ in D. Thus, since $f \in S^{0,2}_{P^1}(D)$, we conclude that $(f - Pu) \in kerP^1 \cap \ker P^* \cap L^2(F_{|D}) \subset C^\infty(F_{|D})$.

Finally, if we prove that the weak boundary values $({}^tC_j^*(f - Pu))_{|\partial D} = 0$ then $(f - Pu) \in \mathfrak{H}^{0,2}(D)$ and, due to condition (2) of the theorem, $Pu = f$ in D.

To this aim we fix a section $v^{(j)} \in C^\infty_{comp}(F_j^*)$ and find a section $v \in \mathcal{D}(E^*_{|\overline{D}})$ such that ${}^tB_j^* v = v^{(j)}$ on ∂D, and ${}^tB_i^* v = 0$ on ∂D, if $i \neq j$ (cf. [17], Lemma 28.2). It is clear that $(*_E^{-1} v) \in W^{p,2}(E_{|D})$. Therefore, using (6.2) we obtain that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} < v^{(j)}, {}^tC_j^*(f - Pu) >_y ds =$$

$$= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \sum_{i=0}^{p-1} < {}^tB_i^* v, {}^tC_i^*(f - Pu) >_y ds =$$

$$= \int_D < {}^tP^* v, (f - Pu) >_y dy = \int_D ((f - Pu), P(*_E^{-1} v))_y dy = 0.$$

The proof of the theorem is complete. □

Corollary 6.6. $H(L^2(F_{|D})) = 0$ *if and only if*

(1) $\overline{Im(P)} \subset domR_Y^{p,0}$;

(2) $\mathfrak{H}^{0,2}(D) = 0$.

Proof. It follows form Propositions 6.4 and 6.5. □

The following proposition clarifies the meaning of decomposition (6.1).

Proposition 6.7. $T_Y = P^\star$ *where* $P^\star : L^2(F_{|D}) \to W^{p,2}(E_{|D})$ *is the adjoint (in the sense of Hilbert spaces) of the operator* P *with respect to the scalar product* $H_p^P(.,.)$ *in* $W^{p,2}(E_{|D})$ *and the standard one in* $L^2(F_{|D})$.

Proof. In fact, we proved this in Proposition 4.9. □

Theorem 6.8. *The following conditions are equivalent:*

(1) $H(L^2(F_{|D})) = 0$;

(2) *there exists a constant* $C > 0$ *such that for every* $g \in S^{0,2}_{P^1}(D)$

$$\|g\|_{L^2(F_{|D})} \leq C \|T_Y g\|_{W^{p,2}(E_{|D})};$$

(3) *there exists a constant* $C > 0$ *such that for every* $g \in S^{0,2}_{P^1}(D)$

$$\|g\|_{L^2(F_{|D})} \le C \|PT_Y g\|_{L^2(F_{|Y})}.$$

Proof. Let $H(L^2(F_{|D})) = 0$. Then $S^{0,2}_{P^1}(D) = Im(P) = \overline{Im(P)}$. Hence, due to Proposition 6.7, the ranges of P and T_Y are closed (see, for example, [5], Theorem 1.1.1), i.e. statement (2) holds.

If (2) holds then the range $Im(T_Y)$ is closed. Therefore, from Proposition 6.7 and Theorem 1.1.1 of [5], the range $Im(P)$ is closed. Moreover (2) and Lemma 5.7 imply that $\mathfrak{H}^{0,2}(D) = 0$, i.e., due to Proposition 6.5, condition (1) is satisfied.

Finally, Lemmata 4.7 and 4.5 imply that $S(T_Y g) = (T_Y g)^+$. In particular this means that

$$H^P_p(T_Y g, T_Y g) = \int_Y (PT_Y g, PT_Y g)_x dx = \|PT_Y g\|^2_{L^2(F_{|Y})}.$$

Therefore Proposition 4.12 implies that (2) and (3) are equivalent. □

7. Applications to a P-Neumann problem

In this section we show how Theorem 4.13 can be used to study a P- Neumann problem associated to elliptic differential operator $P \in do_p(E \to F)$.

As in Section refs.1, $\{B_j\}_{j=0}^{p-1}$ is a Dirichlet system of order $(p-1)$ on ∂D and $\{C_j\}_{j=0}^{p-1}$ the one which is associated to $\{B_j\}_{j=0}^{p-1}$ as in Lemma 2.3.

Problem 7.1. Let $r \ge 0$ and $\psi_j \in W^{r-p+j+1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ be given sections. We want to find $\psi \in W^{p,2}(E_{|D})$ such that

$$\begin{cases} P^* P\psi = 0 & in \quad D \\ {}^t C^*_j P\psi = \psi_j & on \quad \partial D \\ (0 \le j \le p-1), & \quad (P\psi) \in W^{r,2}(F_{|D}). \end{cases}$$

The equation $P^* P\psi = 0$ in D has to be understood in the sense of distributions, while the boundary values are intended in the variational sense :

$$(7.1) \int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) = \int_D (P\psi, Pv)_y dy \text{ for every } v \in C^\infty(E_{|\overline{D}}).$$

In particular we obtain

Proposition 7.2. *A necessary condition in order that* Problem 7.1 *be solvable for given* $\psi_j \in W^{r-p+j+1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ *is that*

$$(7.2) \qquad \int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j}) B_j v, \psi_j >_y ds(y) = 0 \textit{ for every } v \in S^{p,2}_P(D).$$

Proof. Indeed, because $C^\infty(E_{|\overline{D}})$ is dense in $W^{p,2}(E_{|D})$, formula (7.1) extends by continuity to $v \in W^{p,2}(E_{|D})$. □

Proposition 7.3. *Let $\psi_j = 0$ $(0 \le j \le p-1)$. Then $\psi \in W^{p,2}(E_{|D})$ is a solution of* Problem 7.1 *if and only if $\psi \in S^{p,2}_P(D)$.*

Proof. Obviously, a section $\psi \in S^{p,2}_P(D)$ is a solution of Problem 7.1 with $\psi_j = 0$ $(0 \le j \le p-1)$. Conversely, if ψ is a solution of Problem 7.1 with $\psi_j = 0$ $(0 \le j \le p-1)$ then $T_Y P\psi = 0$. Hence $\psi = M_Y\psi = \lim_{\nu\to\infty} M^\nu\psi$, i.e. $\psi \in S^{p,2}_P(D)$. □

The operator P^*P is a elliptic with C^∞ coefficients, and the ranks of the symbols of the boundary operators $({}^tC^*_j)$ are maximal in a neighbourhood of ∂D. Nevertheless, since, in general, the space $S^{p,2}_P(D)$ is not finite dimensional, Proposition 7.3 implies that the boundary value Problem 7.1 may be not elliptic.

In the following theorem we set

$$\widetilde{T_Y}(\oplus\psi_j) = \int_{\partial D} \sum_{j=0}^{p-1} < {}^tB^*_j(y)\Phi(x,y), \psi_j(y) >_y ds(y).$$

Theorem 7.4. Problem 7.1 *is solvable if and only if the series $\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j)$ converges in the $W^{p,2}(E_{|D})$-norm and $P\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j) \in W^{r,2}(F_{|D})$.*

Proof. Let Problem 7.1 be solvable and let $\psi \in W^{p,2}(E_{|D})$ be a solution. Then $\widetilde{T_Y}(\oplus\psi_j)) = T_Y P\psi$, and, due to Theorem 4.13, the series $R_Y P\psi = \sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j)$ converges in the $W^{p,2}(E_{|D})$-norm. Moreover, using Theorem 4.13, we conclude that $P\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j) = P\psi \in W^{r,2}(F_{|D})$.

Back, assume that the series $\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j)$ converges in the $W^{p,2}(E_{|D})$- norm, and that $P\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j) \in W^{r,2}(F_{|D})$. Let us set $\psi = \sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j)$. Then $P^*P\psi = 0$ in D. Hence to prove that ψ is a solution of Problem 7.1 we need only to prove only that ${}^tC^*_j P\psi = \psi_j$ on ∂D $(0 \le j \le p-1)$.

We note now that

$$M\psi = M\sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j) = \sum_{\mu=0}^\infty M^\mu_Y \widetilde{T_Y}(\oplus\psi_j) - \widetilde{T_Y}(\oplus\psi_j) = \psi - \widetilde{T_Y}(\oplus\psi_j).$$

Hence we obtain, using (3.2) and Stokes' formula :

$$\widetilde{T_Y}(\oplus\psi_j) = T_Y P\psi = \widetilde{T_Y}(\oplus {}^tC^* P\psi) \; in \; Y.$$

Finally (4.2) implies that

$$(\psi_j - {}^tC^*_j P\psi)_{|\partial D} = ({}^tC^*_j P\widetilde{T_Y}(\oplus(\psi_j - {}^tC^*_j P\psi))^-)_{|\partial D} -$$

$$-({}^tC^*_j P\widetilde{T_Y}(\oplus(\psi_j - {}^tC^*_j P\psi))^+)_{|\partial D} = 0.$$

Theorem 7.4 is proved. □

Proposition 7.5. *Let* $\psi_j \in W^{-p+j+1/2,2}(F_{j|\partial D})$ $(0 \leq j \leq p-1)$. *If* Problem 7.1 *is solvable then the series*

$$\psi = \sum_{\nu=0}^{\infty} (M_Y)^{\nu} \widetilde{T_Y}(\oplus \psi_j),$$

converging in the $W^{p,2}(E_{|D})$*-norm, is the (unique) solution of* Problem 7.1 *belonging to* $(S^{p,2}(D))^{\perp}$.

Proof. See the proof of Theorem 7.4. □

In the case where $P = \overline{\partial}$ (the Cauchy-Riemann system) in $\mathbb{C}^n$ such a formula was obtained by Kytmanov (see [7], p.177).

In the remaining part of this section we will show how the P-Neumann problem 7.1 connects to the solvability of the equation $Pu = f$ and to the closedness of the image of the operator P.

Let us first investigate criterions for $f \in domR_Y^{p,r}$ (see Theorem reft.4.6). To this purpose we consider the following problem.

Problem 7.6. Given a section $v \in (S^{p,2}(D))^{\perp} \cap W^{r+p,2}(E_{|D})$, find a section $\varphi \in W^{p,2}(E_{|D})$ such that

$$\begin{cases} T_Y P \varphi = v, \\ (P\varphi) \in W^{r,2}(F_{|D}). \end{cases}$$

Theorem 7.7. *Let* $f \in W^{r,2}(F_{|D})$ $(r \geq 0)$. *The following conditions are equivalent:*

(1) $f \in domR_Y^{p,r}$;

(2) *for every* $v \in S^{p,2}_{\Delta}(D)$ *we have*

$$\begin{cases} \int_{-0}^{1} \frac{d_\lambda (H_p^P(E_\lambda T_Y f, v))}{1-\lambda} < \infty, \\ P(\int_{-0}^{1} \frac{dE_\lambda (T_Y f)}{1-\lambda}) \in W^{r,2}(F_{|D}); \end{cases}$$

(3) *The* P*-Neumann* Problem 7.1 *is solvable for* $\{\psi_j = ({}^tC_j^* P(T_Y f)^+)_{|\partial D}\}_{(0 \leq j \leq p-1)}$;

(4) Problem 7.6 *is solvable for* $v = T_Y f$.

Proof. (1) ⇔ (2). The statement follows from the following chain of equalities:

$$\sum_{\mu=0}^{\infty} M_Y^{\mu}(T_Y f) = \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu-1} \int_{-0}^{1} \lambda^{\mu} dE_\lambda (T_Y f) =$$

$$= \lim_{\nu\to\infty} \int_{-0}^{1} \sum_{\mu=0}^{\nu-1} \lambda^{\mu} dE_\lambda (T_Y f) = \lim_{\nu\to\infty} \int_{-0}^{1} \frac{(1-\lambda^{\nu}) dE_\lambda (T_Y f)}{1-\lambda}.$$

(1) $\Leftrightarrow$ (3). Lemma 4.10 and Theorem 4.13 imply that

$$R_Y f = \sum_{\mu=0}^{\infty} M_Y^{\mu} T_Y f = T_Y P T_Y f + \sum_{\mu=0}^{\infty} M_Y^{\mu} T_Y P(S(T_Y f)) =$$

$$= T_Y P T_Y f + \sum_{\mu=0}^{\infty} M_Y^{\mu} T_Y P((T_Y f)^+) = T_Y P T_Y f + \sum_{\mu=0}^{\infty} M_Y^{\mu} \widetilde{T_Y}(\oplus {}^t C_j^* P(T_Y f)^+).$$

This means that the series $\sum_{\mu=0}^{\infty} M_Y^{\mu} \widetilde{T_Y}(\oplus {}^t C_j^* P(T_Y f)^+)$ converges in the $W^{p,2}(E_{|D})$-norm, and $P \sum_{\mu=0}^{\infty} M_Y^{\mu} \widetilde{T_Y}(\oplus {}^t C_j^* P(T_Y f)^+) \in W^{r,2}(F_{|D})$ if and only if $f \in dom R_Y^{p,r}$. Therefore the statement follows from Theorem 7.4.

(1) $\Leftrightarrow$ (4). Let $f \in dom R_Y^{p,r}$ then (5.5) implies that $(f - P R_Y f) \in ker T_Y \cap W^{r,2}(F_{|D})$, that is $\varphi = R_Y f$, because $\ker T_Y = \ker R_Y^{p,r}$ by Proposition 5.4.

Conversely, if (4) holds then Theorem 4.13 implies that the series $R_Y P\varphi = R_Y f$ converges in the $W^{p,2}(E_{|D})$-norm, and $P R_Y f = P R_Y P \varphi = P\varphi \in W^{r,2}(F_{|D})$. Therefore $f \in dom R_Y^{p,r}$.

The proof of Theorem 7.7 is complete. □

Remark 7.8. We emphasize that the NEUMANN Problem 7.1 is the P-NEUMANN problem associated with the differential complex $\{E^i, P^i\}$ (see, for example, [16], p. 136) at step $i = 0$. However, as a rule, in order to solve the equation $Pu = f$, the P-NEUMANN problem was studied in the case $i = 1$.

Proposition 7.9. *Let $u \in S^{p,2}(D)^{\perp}$ and $\psi_j = ({}^t C_j^* P S(u))_{|\partial D}$ $(0 \le j \le p-1)$. Then the necessary condition rm (7.2) for the solvability of* Problem 7.1 *holds, i.e. for every $v \in S_P^{p,2}(D)$ we have*

$$\int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j} B_j v, \psi_j >_y ds(y) = 0.$$

Proof. Indeed, formula (refeq.3.5) implies that

$$\int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j} B_j v, \psi_j >_y ds(y) = \int_{\partial D} \sum_{j=0}^{p-1} < (*_{F_j} B_j v, {}^t C_j^* P S(u)) >_y ds(y) =$$

$$= - \int_{Y \setminus D} (PS(u), PS(v))_y dy = -H_p^P(u, v) = 0$$

for every $v \in S_P^{p,2}(D)$. □

Because $S(T_Y f) = (T_Y f)^+$, Propositions 5.9 and 7.9 imply that condition (7.2) holds for $\psi_j = ({}^t C_j^* P(T_Y f)^+)_{\partial D}$ $(0 \le j \le p-1)$.

Now let us see the connection between the P-NEUMANN problem and the closedness of the range of the operator P.

Proposition 7.10. $\overline{Im(T_Y)} = \overline{Im(T_Y P)} = (S_P^{p,2}(D))^{\perp}$.

Proof. Proposition 5.9 and Lemma 4.7 imply that $Im(T_Y) \subset (S^{p,2}_P(D))^\perp$. Therefore, since $(S^{p,2}_P(D))^\perp$ is a closed subspace of $W^{p,2}(E_{|D})$, $\overline{Im(T_Y P)} \subset \overline{Im(T_Y)} \subset (S^{p,2}_P(D))^\perp$.

Conversely, formula (3.2) and Corollary 3.3 imply that

$$v = \sum_{\nu=0}^{\infty} M_Y^\nu T_Y P v = \sum_{\nu=0}^{\infty} (Id - T_Y P)^\nu T_Y P v$$

for every $v \in (S^{p,2}_P(D))^\perp)$. Therefore $(S^{p,2}_P(D))^\perp \subset \overline{Im(T_Y P)} \subset \overline{Im(T_Y)}$. □

Proposition 7.11. *The range $Im(T_Y)$ is closed if and only if the range $Im(T_Y P)$ is closed.*

Proof. Let $Im(T_Y P)$ be closed. Then, due to Proposition 7.10,

$$(S^{p,2}_P(D))^\perp = \overline{Im(T_Y P)} = Im(T_Y P) \subset Im(T_Y) \subset (S^{p,2}_P(D))^\perp.$$

Hence the inclusions are equivalent and the range $Im(T_Y)$ is closed.

Conversely, if the range $Im(T_Y)$ is closed then Proposition 6.7 and Theorem 1.1.1 of [5] imply that the range $Im(P)$ is closed. Therefore $Im(T_Y P) = \overline{Im(T_Y P)}$ because T_Y is in this case a topological homomorphism. □

Proposition 7.12. *The range $Im(P)$ is closed if and only if the P-*Neumann *Problem 7.1 is solvable for all $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$ satisfying (7.2).*

Proof. Let $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$. Then $\widetilde{T}_Y(\oplus\psi_j) \in W^{p,2}(E_{|D})$ (see [9], 2.3.2.4). Moreover, if $\oplus\psi_j$ satisfies (7.2) then, due to (4.2), we have

$$H^P_p(\widetilde{T}_Y(\oplus\psi_i), v) = \int_{\partial D} \sum_{i=0}^{p-1} < *_{F_i} B_i v, {}^tC_i^* P \widetilde{T}_Y(\oplus\psi_i)^- - {}^tC_i^* P \widetilde{T}_Y(\oplus\psi_i)^+ >_y ds =$$

$$\int_{\partial D} \sum_{i=0}^{p-1} < *_{F_i} B_i v, \psi_i >_y ds = 0$$

for every $v \in S^{p,2}_P(D)$. That is, $\widetilde{T}_Y(\oplus\psi_j) \in (S^{p,2}_P(D))^\perp$. On the other hand, if $Im(P)$ is closed then, according to Propositions 6.7, 7.10, and 7.11, and Theorem 1.1.1 of [5] $Im(T_Y P) = (S^{p,2}_P(D))^\perp$. In particular it means that there exists a section $\varphi \in W^{p,2}(E_{|D})$ such that $T_Y P\varphi = \widetilde{T}_Y(\oplus\psi_j)$. Therefore, from Theorem 4.13 and Corollary 3.3, the series

$$\sum_{\nu=0}^{\infty} M_Y^\nu T_Y P \varphi = \sum_{\nu=0}^{\infty} M_Y^\nu \widetilde{T}_Y(\oplus\psi_j)$$

converges in the $W^{p,2}(E_{|D})$-norm. Now using Theorem 7.4 we conclude that Problem 7.1 is solvable.

Conversely, let $v \in (S^{p,2}_P(D))^{\perp}$. Then $({}^tC^*_j PS(v))_{|\partial D} \in W^{-p+j+1/2,2}(F_{j|\partial D})$, and, due to Proposition 7.9, $(\oplus {}^tC^*_j PS(v))_{|\partial D}$ satisfies (7.2). Hence, if Problem 7.1 is solvable for all $\oplus\psi_j$ satisfying (7.2), there exists a section $\psi \in S^{p,2}_{\Delta}(D)$ such that $(\oplus {}^tC^*_j P\psi)_{|\partial D} = (\oplus {}^tC^*_j PS(v))_{|\partial D}$. In particular, from Lemma 4.10, we have

$$v = T_Y Pv + T_Y PS(v) = T_Y Pv + \widetilde{T_Y}(\oplus {}^tC^*_j PS(v)) = T_Y P(v + \psi).$$

Therefore $Im(T_Y p) = (S^{p,2}_P(D))^{\perp}$, i.e. $ImT_Y P$ is closed, and, due to Propositions 6.7, 7.11 and Theorem 1.1.1 of [5], $Im(P)$ is closed. $\square$

8. Examples

Using Proposition 7.12 we can obtain a result on the solvability of the P-Neumann Problem 7.1 in the case where P is elliptic.

Corollary 8.1. *Let P be an elliptic operator in X such that the operators P and P^*P have bilateral fundamental solutions on X. Then* Problem 7.1 *is solvable for every $\oplus\psi_j \in \oplus W^{-p+j+1/2,2}(F_{j|\partial D})$ satisfying (7.2).*

Proof. According to Corollary 2.6, for every $f \in L^2(F_{|D})$ there exist a $W^{p,2}(E_{|D})$ solution of the equation $Pu = f$. In particular,$Im((P)$ is closed, and therefore, the statement follows from Proposition 7.12. $\square$

We note that in Corollary 8.1 we obtain maximal Sobolev regularity for the solutions of the boundary value Problem 7.1. However the nullspace of the problem may be not finite dimensional (see Proposition 7.3) and hence this may be not an elliptic boundary value problem.

Example 8.2. Let $P = \Delta$ be Laplace operator in $\mathbb{R}^n$. Then $P^*P = \Delta^2$ and hence the operators P and P^*P have bilateral fundamental solutions in X.

Let $D \Subset \mathbb{R}^n$ be a domain with C^∞-smooth boundary ∂D. As a Dirichlet system on ∂D we can take the system $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$. Then, by simple calculations, the system $\{C_0 = -\frac{\partial}{\partial n}, C_1 = 1\}$ is the system associated to $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$ in Lemma 2.3. Therefore Corollary 8.1 implies that the problem

$$\begin{cases} \Delta^2\psi = 0 & in & D, \\ -\frac{\partial}{\partial n}\Delta\psi = \psi_0 & on & \partial D, \\ \Delta\psi = \psi_1 & \text{on } \partial D & \psi \in W^{2,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-3/2,2}(\partial D)$, $\psi_1 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D} (\psi_0(y)v(y) - \psi_1(y)\frac{\partial v}{\partial n}(y))ds(y) = 0 \text{ for all harmonic } W^{2,2}(D) - \text{functions } v.$$

Example 8.3. Let P be the Cauchy - Riemann system on the plane $\mathbb{C}^1 \cong \mathbb{R}^2$, i.e. $P = \frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2}$. In the complex form with $z = x_1 + \sqrt{-1}x_2$, $\overline{z} = x_1 - \sqrt{-1}x_2$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2})$, we have $P = 2\frac{\partial}{\partial \overline{z}}$, $P^* = -2\frac{\partial}{\partial z}$. Then $P^*P = -\Delta$ is the Laplace operator in $\mathbb{R}^2$ and hence the operators P and P^*P have bilateral fundamental solutions on X.

Let $D \Subset \mathbb{R}^2$ be a domain with C^∞-smooth boundary ∂D. As a Dirichlet system on ∂D we can take the system $\{B_0 = 1\}$. Then, setting

$$\begin{cases} \rho(x) = -dist(x, \partial D), & x \in \overline{D}, \\ \rho(x) = dist(x, \partial D), & x \notin \overline{D}, \end{cases}$$

the function ρ belongs to the class of functions defining the domain D ($D = \{x \in X : \rho(x) < 0\}$), $|d\rho| = \sqrt{\sum_{j=1}^{2}(\frac{\partial \rho}{\partial \overline{x}_j})^2} = 1$ in a neighbourhood of ∂D and the system $\{C_0 = 2\frac{\partial \rho}{\partial \overline{z}}\}$ is the system associated to $\{B_0 = 1\}$ in Lemma 2.3. Therefore Corollary 8.1 implies that the problem

$$-\begin{cases} Delta\psi = 0 & in \quad D, \\ 4\frac{\partial \rho}{\partial z}\frac{\partial \psi}{\partial \overline{z}} = \psi_0 & on \quad \partial D, \\ \psi \in W^{1,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-1/2,2}(\partial D)$, satisfying

$$\int_{\partial D} (\psi_0(y)v(y))ds(y) = 0 \text{ for all holomorphic } W^{1,2}(D) - \text{functions } v.$$

The problem above is nothing but the $\overline{\partial}$ -Neumann problem for functions in $\mathbb{C}^1$.
Consider now situation where the operator P is overdetermined (elliptic).

Example 8.4. Let P be the Cauchy - Riemann system in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ ($n > 1$), i.e. $P = \begin{pmatrix} \frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_{n+1}} \\ \cdots \\ \frac{\partial}{\partial x_n} + \sqrt{-1}\frac{\partial}{\partial x_{2n}} \end{pmatrix}$. In the complex form with $z_j = x_j + \sqrt{-1}x_{n+j}$, $\overline{z}_j = x_j - \sqrt{-1}x_{n+j}$, $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial x_{n+j}})$, $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial x_{n+j}})$, we have $P = 2\begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \cdots \\ \frac{\partial}{\partial \overline{z}_n} \end{pmatrix}$ $(= 2\overline{\partial})$, $P^* = -2\begin{pmatrix} \frac{\partial}{\partial z_1} \\ \cdots \\ \frac{\partial}{\partial z_n} \end{pmatrix}$ $(= 2\partial)$. Then $P^*P = -\Delta$ is the Laplace operator in $\mathbb{R}^{2n}$ and hence the operator P^*P has a bilateral fundamental solution in X. However, due to the removability theorem for compact singularities of holomorphic functions in $\mathbb{C}^n$, the Cauchy-Riemann system in $\mathbb{C}^n$ has no right fundamental solution.

It is known that if the domain D is not pseudo-convex then the range $Im(P) : W^{1,2}(D) \to [L^2(D)]^n$ may be not closed. But even in a strictly convex domain D we can not achieve maximal global regularity for solutions of the equation $\overline{\partial}u = f \in [L^2(D)]^n$.

Indeed, let D be the ball $B(0,R)$ in $\mathbb{C}^2$ with centre at 0 and radius $0 < R < \infty$. Then $f = \begin{pmatrix} 0 \\ \frac{1}{R - z_1} \end{pmatrix} \in [L^2(D)]^2$ and the function $u = \frac{\overline{z}_2}{R - z_1} \in L^2(D)$ is a solution of the equation $\overline{\partial}u = f$ in D. Because

$$\frac{\partial u}{\partial z_1} = \frac{\overline{z}_2}{(R - z_1)^2} \notin L^2(D)$$

we conclude that $u \notin W^{1,2}(D)$.

Assume that there exists a function $v \in W^{1,2}(D)$ satisfying $\overline{\partial}v = f$. Then $v = u + h$ where h is a holomorphic L^2-function in the ball D and $\frac{\partial v}{\partial z_1} \in L^2(D)$. Hence

$$\|\frac{\partial v}{\partial z_1}\|^2_{L^2(D)} = lim_{\varepsilon\to 0}\|\frac{\partial v}{\partial z_1}\|^2_{L^2(D_\varepsilon)} = lim_{\varepsilon\to 0}(\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D_\varepsilon)} + \|\frac{\partial h}{\partial z_1}\|^2_{L^2(D_\varepsilon)} +$$

$$-\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z} - \overline{\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z}}) < \infty.$$

On the other hand,

$$-\frac{1}{4}\int_{D_\varepsilon}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})dz\wedge d\overline{z} =$$

$$= \frac{1}{2\sqrt{-1}}\int_{|z_1|\leq R-\varepsilon}\int_{r=0}^{(R-\varepsilon)^2-|z_1|^2}\int_{|z_2|=r}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})\frac{\sqrt{-1}d\overline{z}_2}{\overline{z}_2}rdrdz_1\wedge d\overline{z}_1 = 0$$

because $\frac{1}{\overline{z}_2}\frac{\partial u}{\partial z_1}$, $(\overline{\frac{\partial h}{\partial z_1}})$ are anti-holomorphic with respect to z_2 and hence

$$\int_{|z_2|=r}\frac{\partial u}{\partial z_1}(\overline{\frac{\partial h}{\partial z_1}})\frac{d\overline{z}_2}{\overline{z}_2} = 0 \ (0 < r < R).$$

Therefore we obtain

$$\|\frac{\partial v}{\partial z_1}\|_{L^2(D)} = lim_{\varepsilon\to 0}(\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D_\varepsilon)} + \|\frac{\partial h}{\partial z_1}\|^2_{L^2(D_\varepsilon)}) < \infty$$

contradicting $\|\frac{\partial u}{\partial z_1}\|^2_{L^2(D)} = \infty$.

Thus we proved that for every ball $D = B(0,R) \subset \mathbb{C}^2$ there exists a closed differential $(0,1)$-form f with coefficients in $L^2(D)$ for which there is no $W^{1,2}(D)$-solution of the equation $\overline{\partial}u = f$ (cf. [6] for an analogous result for Hölder spaces).

Now using results of [5] (on triviality of the "harmonic" spaces $\widetilde{\mathfrak{H}}^{0,2}(D)$) and Propositions 7.12 we conclude that the image $Im(\overline{\partial}) : W^{1,2}(D) \to [L^2(D)]^2$ is not closed.

Let ρ be as in Example 8.3 then ρ belongs to the class of functions defining the domain D ($D = \{x \in X : \rho(x) < 0\}$), $|d\rho| = \sqrt{\sum_{j=1}^{2n}(\frac{\partial\rho}{\partial x_j})^2} = 1$ in a neighbourhood of $\partial D \in C^\infty$ and the system $\{C_0 = 2(\frac{\partial\rho}{\partial\overline{z}_1}, \dots, \frac{\partial\rho}{\partial\overline{z}_n})\}$ is the system associated to $\{B_0 = 1\}$ in Lemma 2.3.

Therefore, even if D is a ball, the boundary value problem

$$\begin{array}{ll} -\Delta\psi = 0 & \text{in } D, \\ 4\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}\frac{\partial\psi}{\partial\overline{z}_j} & = \psi_0 \text{ on } \partial D, \end{array}$$

is not solvable in $W^{1,2}(D)$ for all (complex valued) data $\psi_0 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D} (\psi_0(y) v(y)) ds(y) = 0 \text{ for all holomorphic } W^{1,2}(D) - functions\ v.$$

The problem above is nothing but the $\overline{\partial}$ -Neumann problem for functions in $\mathbb{C}^n$. Results about the solvability of this problem could be found, for example, in [7].

It is easier to prove that we can not achieve the maximal global regularity in the case where boundary of D is more "flat". For instance, if D is the bidisk in $\mathbb{C}^2$ with centre at 0 and radius $0 < R < \infty$, then arguing as before one sees that for $f = \begin{pmatrix} 0 \\ \frac{1}{(R-z_1)^\delta} \end{pmatrix} \in [L^2(D)]^2$ $(1/2 < \delta < 1)$ there is no $W^{1,2}(D)$-solution of the equation $\overline{\partial} u = f$ in D.

Example 8.5. Let $X = \mathbb{R}^n$ and $P = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \\ \cdots \\ \frac{\partial^2}{\partial x_n^2} \end{pmatrix}$. Then $P^* P = \sum_{j=1}^n \frac{\partial^4}{\partial x_j^4}$. It is clear that $P^* P$ has a bilateral fundamental solution on X but the operator P has only a left one.

However, it is not difficult to see that in every domain D, where we can find a solution with maximal (global) regularity of the equation $grad(u) = f$ in D, we can also solve with maximal (global) regularity the equation $Pu = f$. For instance, we can do it in every convex domain with $\partial D \in C^2$.

As a Dirichlet system on ∂D we can take the system $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$. If the function ρ is as in Example 8.3, then ρ belongs to the class of functions defining the domain D $(D = \{x \in X : \rho(x) < 0\})$, $|d\rho| = \sqrt{\sum_{j=1}^n (\frac{\partial \rho}{\partial x_j})^2} = 1$ in a neighbourhood of ∂D and the system of boundary differential operators

$$\{C_0 = -(\frac{\partial \rho}{\partial x_1}\frac{\partial}{\partial x_1}, \dots, \frac{\partial \rho}{\partial x_n}\frac{\partial}{\partial x_n}), \qquad C_1 = ((\frac{\partial \rho}{\partial x_1})^2, \dots, (\frac{\partial \rho}{\partial x_n})^2)\}$$

is the system associated to $\{B_0 = 1, B_1 = \frac{\partial}{\partial n}\}$ in Lemma 2.3.

Therefore Proposition 7.12 implies that the Neumann problem

$$\begin{cases} \sum_{j=1}^n \frac{\partial^4}{\partial x_j^4}\psi = 0 & \text{in} \quad D, \\ -\sum_{j=1}^n \frac{\partial \rho}{\partial x_j}\frac{\partial^3 \psi}{\partial x_j^3} = \psi_0 & \text{on} \quad \partial D, \\ \sum_{j=1}^n (\frac{\partial \rho}{\partial x_j})^2 \frac{\partial^2 \psi}{\partial x_j^2} = \psi_1 & \text{on } \partial D, \quad \psi \in W^{2,2}(D) \end{cases}$$

is solvable for all (complex valued) data $\psi_0 \in W^{-3/2,2}(\partial D)$, $\psi_1 \in W^{-1/2,2}(\partial D)$ satisfying

$$\int_{\partial D} (\psi_0(y) v(y) - \psi_1(y) \frac{\partial v}{\partial n}(y)) ds(y) = 0 \text{ for all } S_P^{2,2}(D) - \text{functions } v$$

in every convex domain D with a C^∞-smooth boundary ∂D.

Obviously, $S_P^{2,2}(D)$ consists of all polynomials of the form

$$\sum_{k\neq i} a_{k,i} x_k x_i + \sum_{j=1}^{n} b_j x_j + c$$

where $a_{k,i}, b_j, c \in \mathbb{C}^1$.

9. Applications to the Cauchy and Dirichlet problems

We have proved the solvability of the Dirichlet problem for an determined elliptic operator $P^*P = \Delta \in do_{2p}(E \to E)$ in Lemma 4.2. Let us now obtain a formula for the solution of this problem. In the following proposition $M(\oplus\psi_j)$ stands for the integral

$$M_Y(\oplus\psi_j)(x) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j(y)^t P^*(y)\Phi_Y(x,y), \psi_j >_y ds.$$

Proposition 9.1. *Let the operator P satisfy the Uniqueness Condition in the small on X and ∂D be connected. Then, if $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1, m \ge p)$, the series*

$$\psi = \sum_{\nu=0}^{\infty} (T_Y P)^\nu M_Y(\oplus\psi_j),$$

converging in the $W^{p,2}(E_{|D})$-norm, is the (unique) $W^{m,2}(E_{|D})$-solution of the Dirichlet problem for the operator P^*P and the data ψ_j $(0 \le j \le p-1)$.

Proof. We proved in Lemma 4.2 that for sections $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$ there exists a unique solution $\psi \in W^{m,2}(E_{|D})$ of the Dirichlet problem. Theorem 4.13 and Corollary 3.3 imply that

$$\psi = \lim_{\nu\to\infty}(T_Y P)^\nu \psi + \sum_{\nu=0}^{\infty}(T_Y P)^\nu M\psi =$$

$$= \lim_{\nu\to\infty}(T_Y P)^\nu \psi + \sum_{\nu=0}^{\infty}(T_Y P)^\nu M_Y(\oplus\psi_j).$$

On the other hand, under the hypothesis of the proposition $\widetilde{S}_P^{p,2}(D) = W_0^{p,2}(E_{|D})$ (see Remark 4.14), and therefore $\lim_{\nu\to\infty}(T_Y P)^\nu\psi = 0$, i.e.

$$\psi = \sum_{\nu=0}^{\infty} (T_Y P)^\nu M_Y(\oplus\psi_j),$$

which was to be proved. □

This formula may be useful in cases where the Green function is known for a large domain Y (for instance where Y is a ball in $\mathbb{R}^n$, Δ is the usual Laplace operator and

$D \Subset Y$ is a domain with connected boundary, for which the Green function is not known).

We consider now the Cauchy problem for the operator P.

Problem 9.2. Let S be an open connected subset of ∂D and $\psi_j \in W^{m-j-1/2,2}(F_{j|\overline{S}})$ $(0 \le j \le p-1, m \ge p)$ be given sections. It is required to find a section $\psi \in W^{m,2}(E_{|D})$ such that

$$\begin{cases} P\psi = 0 & \text{in} \quad D \\ B_j\psi = \psi_j & \text{on} \quad \partial D \\ (0 \le j \le p-1). & \end{cases}$$

We obtain a solvability condition for the Cauchy problem in the degenerate case where the Cauchy data are given on the whole boundary, i.e. $S = \partial D$.

Proposition 9.3. *Let $u \in W^{m,2}(E_{|D})$. The following conditions are equivalent:*

(1) $u \in S_P^{m,2}(D)$;

(2) $M_Y u = u$ *in* D;

(3) $T_Y P u = 0$ *in* D.

Proof. Formula (3.2) implies that (2) and (3) are equivalent. Let $M_Y u = u$ in D then, due to Theorem 4.13 $u = (\lim_{\nu\to\infty} M_Y^\nu) \in S_P^{p,2}(D)$. Since $u \in W^{m,2}(E_{|D})$ we conclude that $u \in S_P^{m,2}(D)$. □

Proposition 9.4. *Let $u \in S_\Delta^{m,2}(D)$. The following conditions are equivalent:*

(1) $u \in S_P^{m,2}(D)$;

(2) $M_Y u = 0$ *in* $Y\backslash\overline{D}$;

(3) $T_Y P u = 0$ *in* $Y\backslash\overline{D}$.

Proof. Formula (3.2) implies that (2) and (3) are equivalent. Let $T_Y P u = 0$ in $Y\backslash\overline{D}$ then, $TP_Y u = 0$ in D and the statement follows from Proposition 9.3. □

Corollary 9.5. *Let $\psi_j \in W^{m-j-1/2,2}(F_{j|\partial D})$ $(0 \le j \le p-1)$. Then Problem 9.2 is solvable if and only if $M_Y(\oplus\psi_j) = 0$ in $Y\backslash\overline{D}$.*

Proof. If Problem 9.2 is solvable and $\psi \in S_P^{m,2}(D)$ is the solution then $M_Y(\oplus\psi_j) = M_Y\psi$. Using Theorem 2.4 we conclude that $M_Y(\oplus\psi_j) = 0$ in $Y\backslash\overline{D}$.

Conversely, if $M_Y(\oplus\psi_j) = 0$ in $Y\backslash\overline{D}$ then (4.2) mplies that

$$(B_j M_Y(\oplus\psi_j)^-)_{|\partial D} =$$

$$(B_j M_Y(\oplus\psi_j)^-)_{|\partial D} - (B_j M_Y(\oplus\psi_j)^+)_{|\partial D} = \psi_j \ (0 \le j \le p-1). \tag{9.1}$$

We set now $\psi = M_Y(\oplus\psi_j)^-$. The Theorem on boundedness for potential (coboundary) operators in Sobolev spaces (see [9], 2.3.2.5) implies that $\psi \in S_\Delta^{m,2}(D)$. On the other hand (9.1) implies that $M_Y\psi = M_Y(\oplus\psi_j)$, i.e. $M_Y\psi = 0$ in $Y\setminus\overline{D}$. Therefore the statement follows from Proposition 9.4. $\square$

For the Cauchy-Riemann system and the Martinely-Bochner integral Corollary 9.5 was obtained by Kytmanov (see [7], p. 170), and for matrix factorizations of the Laplace operator in $\mathbb{R}^n$ it was proved by one of the authors (see [14]).

In [15] necessary and sufficient conditions for the solvability of the Cauchy Problem 9.2 were obtained in terms of the Green operator M in the case where the coefficients of the operator P are real analytic or, if P is determined elliptic, where the Uniqueness Condition in the small on X holds for the operator P (see Remark 4.14).

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