Green Integrals on Manifolds with Cracks

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Abstract

We prove the existence of a limit in $H^m(D)$ of iterations of a double layer potential constructed from the Hodge parametrix on a smooth compact manifold with boundary, X, and a crack $S \subset \partial D$, D being a domain in X. Using this result we obtain formulas for Sobolev solutions to the Cauchy problem in D with data on S, for an elliptic operator A of order $m \geq 1$, whenever these solutions exist. This representation involves the sum of a series whose terms are iterations of the double layer potential. A similar regularisation is constructed also for a mixed problem in D.

Contents

1 Introduction

This paper is based on the following simple observation. Consider an operator equation $Au = f$ with a bounded operator $A : H_0 \to H_1$ in Hilbert spaces. Suppose each element $u \in H_0$ can be written in the form $u = \pi_0 u + \pi_1 Au$ where π_0 is a projection onto the kernel of A in H_0. Then it is to be expected that under reasonable conditions the element $\pi_1 f$ defines a solution to the equation $Au = f$.

For the Cauchy-Riemann operator $A = \bar{\partial}$ in $\mathbb{C}^n$, $n > 1$, the double layer potential involved in the regularisation formula is just the Martinelli-Bochner integral. In this case, results similar to ours were obtained by Romanov [6].

Theorem 1.1 *Let D be a bounded domain in $\mathbb{C}^n$, $n > 1$, with a connected boundary of class C^1, and Mu stand for the Martinelli-Bochner integral of a function $u \in H^1(D)$. Then the limit $\lim_{N\to\infty} M^N$ exists in the strong operator topology of $H^1(D)$, and it is equal to π_0, a projection onto the (closed) subspace of holomorphic functions in $H^1(D)$.*

By using this result Romanov [6] obtained an explicit formula for a solution $u \in H^1(D)$ to $\bar{\partial} u = f$, where D is a pseudoconvex domain with a smooth boundary, and f a $\bar{\partial}$-closed $(0,1)$-form with coefficients in $H^1(D)$.

2 Preliminary results

Let X be a C^∞ manifold of dimension n with a smooth boundary ∂X. We tacitly assume that it is enclosed into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles E and F over X, we write $\mathrm{Diff}^m(X; E, F)$ for the space of all linear partial differential operators of order $\leq m$ between sections of E and F.

Denote E^* the conjugate bundle of E. Any Hermitian metric $(.,.)_x$ on E gives rise to a sesquilinear bundle isomorphism $*_E : E \to E^*$ by the equality $\langle *_E v, u\rangle_x = (u, v)_x$ for all sections u and v of E.

Pick a volume form dx on X, thus identifying dual and conjugate bundles. For $A \in \mathrm{Diff}^m(X; E, F)$, denote by $A' \in \mathrm{Diff}^m(X; F^*, E^*)$ the transposed operator and by $A^* \in \mathrm{Diff}^m(X; F, E)$ the formal adjoint operator. We obviously have $A^* = *_E^{-1} A' *_F$, cf. [9, 4.1.4] and elsewhere.

For an open set $O \subset X$, we write $L^2(O, E)$ for the Hilbert space of all measurable sections of E over O with a finite norm $(u, u)_{L^2(O,E)} = \int_O (u, u)_x dx$. When no confusion can arise, we also denote $H^m(O, E)$ the Sobolev space of distribution sections of E over O, whose weak derivatives up to order m belong to $L^2(O, E)$.

Given any open set O in $\overset{\circ}{X}$, the interior of X, we let $\mathcal{S}_A(O)$ stand for the space of weak solutions to the equation $Au = 0$ in O. We also denote by $\mathcal{S}_A^m(O)$ the closed subspace of $H^m(O, E)$ consisting of all weak solutions to $Au = 0$ in O.

Write $\sigma(A)$ for the principal homogeneous symbol of order m of the operator A, $\sigma(A)$ living on the cotangent bundle T^*X of X. From now on we assume that $\sigma(A)$ is injective away from the zero section of T^*X. Hence it follows that the Laplacian $\Delta = A^*A$ is an elliptic differential operator of order $2m$ on X.

Let σ be a compact subset in $\overset{\circ}{X}$. In fact, we assume that σ lies on a smooth closed hypersurface S in X. Our goal will be to construct the Hodge theory of the Dirichlet problem for the Laplacian Δ on the manifold $V = \overset{\circ}{X} \setminus \sigma$ with a crack along σ.

Crack problems are usually treated in the framework of analysis on manifolds with edges, cf. Schulze [7]. One thinks of the boundary of σ on S as an edge of V, the cross-section being a 2-dimensional plane with a cut along a ray. The relevant function spaces are therefore weighted Sobolev spaces $H^{s,w}((V, \partial\sigma), E)$ of smoothness s and weight w, both s and w being real numbers. Recall that if $s \in \mathbb{Z}_+$ it coincides with the completion of sections of E over V, C^∞ up to the boundary and vanishing near $\partial\sigma$, with respect to the norm

$$\|u\|_{H^{s,w}((V,\partial\sigma),E)} = \left(\sum_\nu \int \sum_{|\alpha| \le s} \operatorname{dist}(x, \partial\sigma)^{2(|\alpha|-w)} |D^\alpha(\varphi_\nu u)|^2 dx \right)^{1/2},$$

where (φ_ν) is a partition of unity subordinate to a suitable finite open covering (O_ν) of X.

However, we will deal with the very particular case $H^{m,m}((V, \partial\sigma), E)$ which allows us to restrict ourselves to the usual Sobolev spaces on X.

Namely, let $H^m((V, \partial\sigma), E)$ be the closure of all sections of E over V, C^∞ up to the boundary and vanishing close to $\partial\sigma$, in $H^m(V, E)$.

Theorem 2.1 *If the boundary of σ is smooth, then $H^{m,m}((V, \partial\sigma), E)$ and $H^m((V, \partial\sigma), E)$ coincide as topological vector spaces.*

Proof. Obviously, it is sufficient to show that the $H^{m,m}((V, \partial\sigma), E)$- and $H^m(V, E)$-norms are equivalent on sections of E over V, C^∞ up to the boundary and vanishing close to $\partial\sigma$. Without loss of generality we can consider those sections u whose supports are contained in the domain O_ν of some chart on X.

If O_ν does not meet $\partial\sigma$ then $\operatorname{dist}(x, \partial\sigma)$ is strictly positive in O_ν. Hence the $H^{m,m}((V, \partial\sigma), E)$- and $H^m(V, E)$-norms are equivalent on sections of E with a support in O_ν.

In the case $O_\nu \cap \partial\sigma \neq \emptyset$ we choose local coordinates $x = (x_1, \dots, x_n)$ in O_ν, such that $O_\nu \cap \sigma$ is the half-plane $\{x_n = 0,\ x_{n-1} \le 0\}$. Write $x = (x', x_{n-1}, x_n)$ where $x' = (x_1, \dots, x_{n-2})$. We restrict ourselves to sections $u = u(x', x_{n-1}, x_n)$ supported in $Q \times B$, with Q a rectangle in $\mathbb{R}^{n-2}$, and B a disk with centre 0 and radius $R \gg 1$.

Since

$$\|u\|^2_{H^{m,m}((V,\partial\sigma),E)} = \int \sum_{|\alpha| \le m} \operatorname{dist}(x, \partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx,$$

the $H^m(V,E)$-norm is obviously dominated by the $H^{m,m}((V,\partial\sigma),E)$-norm whence

$$H^{m,m}((V,\partial\sigma),E) \hookrightarrow H^m((V,\partial\sigma),E).$$

On the other hand, the summands involving the derivatives of order m in the norms $\|u\|_{H^{m,m}((V,\partial\sigma),E)}$ and $\|u\|_{H^m(V,E)}$ coincide. To handle lower order summands, we fix a multi-index $\alpha \in \mathbb{Z}^n_+$ with $0 \le |\alpha| \le m-1$. Introduce polar coordinates

$$\begin{cases} x_{n-1} &= r\cos\varphi, \\ x_n &= r\sin\varphi \end{cases}$$

in B, and set $U(r) = D^\alpha u\,(x', r\cos\varphi, r\sin\varphi)$. Then

$$\int \operatorname{dist}(x,\partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx = \int_Q dx' \int_{-\pi}^{\pi} d\varphi \int_0^R |r^{|\alpha|-m} U(r)|^2\, r dr.$$

We next make use of a Hardy-Littlewood inequality for measurable functions on the semiaxis with values in a normed space. Namely,

$$\|r^{p-1} \int_0^r f(\varrho) d\varrho\|_{L^q(\mathbb{R}_+)} \le \left(\frac{1}{q'} - p\right)^{-1} \|r^p f(r)\|_{L^q(\mathbb{R}_+)},$$

where $1 \le q \le \infty$, $1/q + 1/q' = 1$ and $p < 1/q'$. Take $f(r) = (\partial/\partial r)U(r)$ and observe that

$$\begin{aligned} |f'(r)| &= |D^{\alpha+1_{n-1}} u\, \cos\varphi + D^{\alpha+1_n} u\, \sin\varphi| \\ &\le |D^{\alpha+1_{n-1}} u| + |D^{\alpha+1_n} u|, \end{aligned}$$

1_j being the multi-index from $\mathbb{Z}^n_+$ which is 1 in the j-th place and 0 in each other one. Repeated application of the Hardy-Littlewood inequality therefore yields

$$\int \operatorname{dist}(x,\partial\sigma)^{2(|\alpha|-m)} |D^\alpha u|^2 dx \le c\, \|D^\alpha u\|^2_{H^{m-|\alpha|}(V,E)},$$

with c a constant independent of u.

Summarising we conclude that the $H^{m,m}((V, \partial\sigma), E)$-norm is majorised by the $H^m(V, E)$-norm on functions vanishing near $\partial\sigma$. This completes the proof. $\square$

More generally, given an open set $O \subset X$ and a closed set $\sigma \subset X$, we denote $H^m((O, \sigma), E)$ the closure of all sections of E over O, C^∞ up to the boundary and vanishing near σ, in $H^m(O, E)$. If $\sigma = \partial O$, we obtain what is usually referred to as

$$\overset{\circ}{H}{}^m(O, E).$$

Fix a Dirichlet system B_j, $j = 0, 1, \ldots, m-1$, of order $m-1$ on the boundary of V. More precisely, each B_j is a differential operator of type $E \to F_j$ and order $m_j \le m-1$ in a neighbourhood U of $\partial X \cup S$. Moreover, the symbols $\sigma(B_j)$, if restricted to the conormal bundle of $\partial X \cup S$, have ranks equal to the dimensions of F_j.

Set $t(u) = \oplus_{j=0}^{m-1} B_j u$, for $u \in H^m(V, E)$. It follows from the results of Hedberg [1] that

$$\overset{\circ}{H}{}^m(V, E) = \{u \in H^m(X, E) : \ t(u) = 0 \ \text{ on } \ \partial X \cup \sigma\}, \tag{2.1}$$

$\partial X \cup \sigma$ being the boundary of V.

Corollary 2.2 *Suppose $\partial\sigma$ is smooth. Then we have a topological isomorphism*

$$\overset{\circ}{H}{}^m(V, E) \cong \{u \in H^{m,m}((V, \partial\sigma), E) : \ t(u) = 0 \ \ \textit{on} \ \ \partial X \cup \overset{\circ}{\sigma}\},$$

the space on the right-hand side being endowed with the norm induced from $H^{m,m}((V, \partial\sigma), E)$.

Proof. By Theorem 2.1 it suffices to show that $\overset{\circ}{H}{}^m(V, E)$ consists of all $u \in H^m((V, \partial\sigma), E)$ such that $t(u) = 0$ on ∂V.

On the one hand, if $u \in \overset{\circ}{H}{}^m(V, E)$ then $u \in H^m((V, \partial\sigma), E)$ and $t(u) = 0$ on ∂V, as is easy to see.

On the other hand, if $u \in H^m((V, \partial\sigma), E)$ and $t(u) = 0$ on ∂V then $u \in H^m((V, \partial V), E)$, as follows from [1]. This just amounts to the desired assertion. $\square$

3 Hodge theory on manifolds with cracks

Let $H^{-m}(V, E)$ denote the dual space of $\overset{\circ}{H}{}^m(V, E)$ with respect to the pairing in $L^2(V, E)$. This is not a canonical definition, we rather follow the notation of [9, 1.4.9].

For every $u \in H^m(V,E)$, the correspondence

$$v \mapsto \int_V (Au, Av)_x \, dx$$

is a continuous conjugate linear functional on $\overset{\circ}{H}{}^m(V,E)$. Thus, the Laplacian $\Delta = A^*A$ extends to a mapping $H^m(V,E) \to H^{-m}(V,E)$.

The following boundary value problem is a straightforward generalisation of the classical Dirichlet problem, cf. [9, 9.2.4].

Problem 3.1 *Given an $F \in H^{-m}(V,E)$, find a section $u \in H^m(X,E)$ such that*

$$\left\{ \begin{array}{rcll} \Delta u & = & F & \text{in } V, \\ t(u) & = & 0 & \text{on } \partial V. \end{array} \right.$$

Another way of stating the problem is to say, "Study the restriction of Δ to $\overset{\circ}{H}{}^m(V,E)$."

If $u \in \overset{\circ}{H}{}^m(V,E)$ and $\Delta u = 0$, then $Au = 0$ in V. In the sequel, $\mathcal{H}(V)$ stands for

$$\overset{\circ}{H}{}^m(V,E) \cap \mathcal{S}_A(V).$$

Furthermore, we let $\mathcal{H}^\perp(V)$ consist of all sections $F \in H^{-m}(V,E)$ satisfying

$$\int_V (F, v)_x \, dx = 0$$

for any $v \in \mathcal{H}(V)$.

Lemma 3.2 *Problem 3.1 is Fredholm. The difference of any two solutions lies in $\mathcal{H}(V)$. The problem is solvable if and only if $F \in \mathcal{H}^\perp(V)$. Moreover, there is a constant $c > 0$ such that for any solution $u \in \mathcal{H}^\perp(V)$ to Problem 3.1, we have*

$$\|u\|_{H^m(X,E)} \leq c \, \|F\|_{H^{-m}(V,E)}. \tag{3.1}$$

Proof. By definition, the equality $\Delta u = F$ means that

$$\int_V (Au, Av)_x \, dx = \int_V (F, v)_x \, dx \tag{3.2}$$

for all $v \in \overset{\circ}{H}{}^m(V,E)$. We are thus looking for a section $u \in \overset{\circ}{H}{}^m(V,E)$ satisfying (3.2).

It readily follows from (3.2) that the null-space of Problem 3.1 is just $\mathcal{H}(V)$. Since

$$\overset{\circ}{H}{}^m(V,E) \hookrightarrow H^m(X,E)$$

and σ is a set of zero measure in $\mathring{X}$, we deduce that

$$\mathcal{H}(V) \hookrightarrow \mathring{H}^m(\mathring{X}, E) \cap \mathcal{S}_A(\mathring{X}),$$

the space on the right-hand side being $\mathcal{H}(\mathring{X})$. Taking into account that the boundary of X is smooth, we deduce that $\mathcal{H}(V)$ is a finite-dimensional subspace of $C^\infty(X, E)$.

That the condition $F \in \mathcal{H}^\perp(V)$ is necessary for the problem to be solvable, follows from (3.2) immediately. Let us prove the sufficiency.

To this end, we invoke the classical Gårding inequality. Namely, as A has injective symbol, we have

$$\|u\|^2_{H^m(X,E)} \le C \int_X (Au, Au)_x \, dx + c \, \|u\|^2_{L^2(X,E)} \tag{3.3}$$

for all $u \in \mathring{H}^m(V, E)$, the constants C and c being independent of u (cf. for instance [10]).

A familiar argument shows that there is a constant $C > 0$ with the property that

$$\|u\|^2_{H^m(X,E)} \le C \int_X (Au, Au)_x \, dx,$$

for each $u \in \mathring{H}^m(V, E) \cap \mathcal{H}^\perp(V)$. Indeed, we argue by contradiction. If there is no such constant then we can find a sequence (u_ν) in $\mathring{H}^m(V, E) \cap \mathcal{H}^\perp(V)$, such that

$$\begin{array}{rcl} \|u_\nu\|_{H^m(X,E)} & = & 1, \\ \|Au_\nu\|_{L^2(X,F)} & < & 2^{-\nu}. \end{array}$$

As the unit ball in a separable Hilbert space is weakly compact, we can assume that (u_ν) converges weakly to a section $u_\infty \in \mathring{H}^m(V, E) \cap \mathcal{H}^\perp(V)$. It follows that

$$\begin{array}{rcl} \displaystyle\int_X (u_\infty, A^*v)_x \, dx & = & \displaystyle\lim_{\nu\to\infty} \int_X (u_\nu, A^*v)_x \, dx \\ & = & \displaystyle\lim_{\nu\to\infty} \int_X (Au_\nu, v)_x \, dx \\ & = & 0 \end{array}$$

for all $v \in C^\infty_{\mathrm{comp}}(\mathring{X}, E)$, i.e. $u_\infty \in \mathcal{H}(V)$. We thus conclude that $u_\infty = 0$. But the Gårding inequality yields

$$1 \le C \, 2^{-\nu} + c \, \|u_\nu\|_{L^2(X,E)}$$

for all ν. Since the inclusion $\overset{\circ}{H}{}^m(V,E) \hookrightarrow L^2(X,E)$ is compact, and thus u_ν converges strongly to u_∞ in $L^2(X,E)$, we get

$$\|u_\infty\|_{L^2(X,E)} \geq 1/c,$$

which contradicts $u_\infty = 0$.

We have thus proved that the Hermitian form

$$\int_X (Au, Av)_x \, dx$$

defines a scalar product in the Hilbert space $\overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$, the corresponding norm being equivalent to the original one. Now the Riesz Theorem enables us to assert that for every $F \in H^{-m}(V,E)$ there exists a unique section

$$u \in \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$$

satisfying

$$\int_V (F, v)_x \, dx = \int_X (Au, Av)_x \, dx$$

for all $v \in \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V)$.

Obviously, every $v \in \overset{\circ}{H}{}^m(V,E)$ can be written in the form $v = v_1 + v_2$, with

$$\begin{array}{rcl} v_1 & \in & \mathcal{H}(V), \\ v_2 & \in & \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V). \end{array}$$

It follows that if $F \in \mathcal{H}^\perp(V)$ then u satisfies (3.2) for all $v \in \overset{\circ}{H}{}^m(V,E)$, as desired.

Finally, since for any section $F \in H^{-m}(V,E)$ "orthogonal" to $\mathcal{H}(V)$ there is a unique solution to Problem 3.1 in

$$\overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V),$$

the estimate (3.1) follows from the Open Map Theorem.

□

We are now in a position to derive a Hodge decomposition for the Dirichlet problem in V.

Theorem 3.3 *There are bounded linear operators*

$$\begin{array}{rrcl} H: & H^{-m}(V,E) & \to & \mathcal{H}(V), \\ G: & H^{-m}(V,E) & \to & \overset{\circ}{H}{}^m(V,E) \cap \mathcal{H}^\perp(V) \end{array}$$

such that

1) *H is the $L^2(V,E)$-orthogonal projection onto the space $\mathcal{H}(V)$, with a kernel $K_H(x,y) = \sum_j h_j(x) \otimes *_E h_j(y)$ where (h_j) is an orthogonal basis of $\mathcal{H}(V)$;*

2) *$AH = 0$ and $GH = HG = 0$;*

3)
$$\begin{array}{rcll} G\Delta u & = & u - Hu & \text{for all } \ u \in \overset{\circ}{H}{}^m(V,E), \\ \Delta GF & = & F - HF & \text{for all } \ F \in H^{-m}(V,E). \end{array}$$

Proof. As already mentioned in the proof of Lemma 3.2, $\mathcal{H}(V)$ is a finite-dimensional subspace of $C^\infty(X,E)$. Denote H the $L^2(V,E)$-orthogonal projection onto $\mathcal{H}(V)$. Fix an orthogonal basis (h_j) for $\mathcal{H}(V)$. Then H has the kernel
$$K_H(x,y) = \sum_j h_j(x) \otimes *_E h_j(y),$$
because
$$(HF)(x) = \sum_j \left(\int_V (F(y), h_j(y))_y \, dy \right) h_j(x)$$
for all $F \in L^2(V,E)$. Since H is a smoothing operator it extends to all of $H^{-m}(V,E)$, too, by
$$(HF)(x) = \langle K_H(x,\cdot), F \rangle_V,$$
for $x \in V$. Clearly,
$$H: \ H^{-m}(V,E) \to \mathcal{H}(V) \hookrightarrow \overset{\circ}{H}{}^m(V,E)$$
is bounded and $AH = 0$.

Pick $F \in H^{-m}(V,E)$. Since $K_H(x,y)^* = K_H(y,x)$ we get
$$\begin{array}{rcl} \displaystyle\int_V (F - HF, v)_x \, dx & = & \displaystyle\int_V (F - HF, Hv)_x \, dx \\ & = & \displaystyle\int_V \left(HF - H^2F, v\right)_x \, dx \\ & = & \displaystyle\int_V (HF - HF, v)_x \, dx \\ & = & 0 \end{array}$$
for all $v \in \mathcal{H}(V)$, i.e.,
$$F - HF \in \mathcal{H}^\perp(V).$$
Therefore, Lemma 3.2 implies that there exists a solution $u \in \overset{\circ}{H}{}^m(V,E)$ to $\Delta u = F - HF$ in V. Setting
$$GF = u - Hu$$

we obtain

$$F = HF + \Delta GF$$

for all $F \in H^{-m}(V, E)$.

As $u - Hu \in \mathcal{H}^{\perp}(V)$ we see from (3.1) that

$$G : \, H^{-m}(V, E) \to \overset{\circ}{H}{}^{m}(V, E) \cap \mathcal{H}^{\perp}(V)$$

is bounded. By definition, $HGF = Hu - H^2 u = 0$ and $GHF = 0$.

On the other hand, we easily obtain the $L^2(V, E)$ -orthogonal decomposition

$$\begin{aligned} u &= Hu + (u - Hu) \\ &= Hu + G\Delta u \end{aligned}$$

for all $u \in \overset{\circ}{H}{}^{m}(V, E)$. This completes the proof.

□

When restricted to $L^2(V, E)$, the operator G is selfadjoint. In fact, given any $F, v \in H^{-m}(V, E)$, we have

$$\begin{aligned} (GF, v) &= (GF, Hv + \Delta Gv) \\ &= (GF, \Delta Gv) \\ &= (\Delta GF, Gv) \\ &= (F, Gv)\,, \end{aligned}$$

$(\cdot, \cdot)$ meaning the scalar product in $L^2(V, E)$. Hence it follows that the Schwartz kernel of G,

$$K_G(\cdot, \cdot) \in \overset{\circ}{H}{}^{m}(V, E) \otimes \overset{\circ}{H}{}^{m}(V, E^*) \hookrightarrow \mathcal{D}'(V \times V, E \otimes E^*),$$

is Hermitian, i.e., $K_G(x, y)^* = K_G(y, x)$ for all $x, y \in V$.

Lemma 3.4 *The operator $T = GA^*$ extends to a continuous linear mapping*

$$L^2(V, F) \to \overset{\circ}{H}{}^{m}(V, E).$$

Proof. For any fixed $f \in L^2(V, F)$, the integral

$$\int_V (f, Av)_x \, dx$$

defines a continuous linear functional on $\overset{\circ}{H}{}^{m}(V, E)$. Hence, the (formal) adjoint A^* extends to a mapping $L^2(V, F) \to H^{-m}(V, E)$, which is obviously continuous. Since G maps $H^{-m}(V, E)$ continuously to $\overset{\circ}{H}{}^{m}(V, E)$, the lemma follows. □

As is easy to check by Stokes' formula, the Schwartz kernel of T is

$$K_T(x,y) = (A^*(y,D))' K_G(x,y),$$

the 'prime' meaning the transposed operator.

Using T, we may rewrite the Hodge decomposition of Theorem 3.3 in the form

$$u = Hu + TAu \tag{3.4}$$

over V, for each $u \in \overset{\circ}{H}{}^m(V,E)$.

We now introduce the Hermitian form

$$h(u,v) = \int_V (Hu, Hv)_x \, dx + \int_V (Au, Av)_x \, dx$$

defined for $u, v \in \overset{\circ}{H}{}^m(V,E)$

Theorem 3.5 *The Hermitian form $h(\cdot,\cdot)$ is a scalar product in $\overset{\circ}{H}{}^m(V,E)$ defining a norm equivalent to the original one. The operator H is also an orthogonal projection from $\overset{\circ}{H}{}^m(V,E)$ onto $\mathcal{H}(V)$ with respect to $h(\cdot,\cdot)$. Moreover,*

$$h(Tf, u) = \int_V (f, Au)_x \, dx$$

for all $f \in L^2(V,F)$ and $u \in \overset{\circ}{H}{}^m(V,E)$.

Proof. The coefficients of A are C^∞ up to the boundary of X, and so $Au \in L^2(V,F)$ for all $u \in \overset{\circ}{H}{}^m(V,E)$. Moreover, it follows from (3.4) that $h(u,u) = 0$ implies $u \equiv 0$ in X. Hence $h(\cdot,\cdot)$ is a scalar product on $\overset{\circ}{H}{}^m(V,E)$.

Since H is a smoothing operator, the original norm of $\overset{\circ}{H}{}^m(V,E)$ is not weaker than $\sqrt{h(\cdot,\cdot)}$.

Further, (3.4) and Lemma 3.4 show that there exists a constant $c > 0$ such that

$$\|u\|_{H^m(V,E)} \le c \left(\|Hu\|_{H^m(V,E)} + \|Au\|_{L^2(V,F)} \right)$$

for all $u \in \overset{\circ}{H}{}^m(V,E)$.

On the other hand, since H is a finite rank operator, there is a constant $C > 0$ such that

$$\|Hu\|_{H^m(V,E)} \le C \, \|Hu\|_{L^2(V,E)}$$

for all $u \in \overset{\circ}{H}{}^m(V,E)$. This proves the equivalence of the topologies.

Suppose $f \in C^\infty_{\mathrm{comp}}(V,F)$ and $u \in \overset{\circ}{H}{}^m(V,E)$. By Theorem 3.3, we get $HTf = 0$. Moreover,

$$\begin{aligned} \int_V (HA^* f, v)_x \, dx &= \int_V (f, AHv)_x \, dx \\ &= 0 \end{aligned}$$

for all $v \in L^2(V,E)$, whence $HA^* f = 0$. Thus,

$$\begin{aligned} h(Tf, u) &= \int_V (AG(A^* f), Au)_x \, dx \\ &= \int_V (\Delta G(A^* f), u)_x \, dx \\ &= \int_V (A^* f - H(A^* f), u)_x \, dx \\ &= \int_V (f, Au)_x \, dx. \end{aligned}$$

As $C^\infty_{\mathrm{comp}}(V,F)$ is dense in $L^2(V,F)$, we obtain the desired assertion on the integral T.

Finally, for any $u, v \in \overset{\circ}{H}{}^m(V,E)$, we have

$$\begin{aligned} h(Hu, &= h(u,v) - h(TAu, v) \\ &= h(u,v) - \int_V (Au, Av)_x \, dx \\ &= \int_V (Hu, Hv)_x \, dx, \end{aligned}$$

i.e., H is a selfadjoint operator in $\overset{\circ}{H}{}^m(V,E)$ with respect to the scalar product $h(\cdot,\cdot)$, and $H^2 = H$, as desired.

$\square$

4 Green formulas on manifolds with cracks

In this section we discuss Green formulas for sections of E on open subsets of V. To this end, we choose a Green operator $G_A(\cdot,\cdot)$ for A on X, cf. [9, 9.2.1]. Given an oriented hypersurface $S \subset X$, we denote $[S]^A$ the kernel over $X \times X$ defined by

$$\left\langle [S]^A, g \otimes u \right\rangle_{X \times X} = \int_S G_A(g, u)$$

for all $g \in C^\infty(X, F^*)$ and $u \in C^\infty(X, E)$ whose supports meet each other in a compact set.

In particular, the kernel $[\partial V]^A$ is supported by the hypersurface $\partial X \cup \sigma$. However, σ, if regarded as a part of the boundary of V, has two sides in X with opposite orientations. When applied to sections g and u whose derivatives up to order $m-1$ are continuous in a neighbourhood of σ, the kernel $[\partial V]^A$ does not include any integration over σ because the integrals over the sides with opposite orientations cancel. In general, the continuity up to the boundary in V does not assume that the limit values from both sides of σ coincide in the interior of σ on S. Hence, $[\partial V]^A$ actually includes, along with the integral over ∂X, the integral over σ of the difference of the limit values of $G_A(g,u)$ on S.

Away from the singularities of V, i.e., $\partial\sigma$, the Green operator G behaves like the Green function of an elliptic boundary value problem, cf. [5]. The edge $\partial\sigma$ is well known to cause additional singularities of the kernel of G on $(V \times \partial\sigma) \cup (\partial\sigma \times V)$.

Given any section $u \in H^m(V,E)$ vanishing in a neighbourhood of $\partial\sigma$, we set

$$\begin{array}{rcl} (Mu)(x) & = & -GA^* \left([\partial V]^A u\right) \\ & = & -\displaystyle\int_{\partial V} G_A(K_T(x,y), u(y)) \end{array}$$

for $x \in V$.

Theorem 4.1 *As defined above, the operator M extends to a continuous mapping of $H^m(V,E)$, and*

$$u = Hu + TAu + Mu \tag{4.1}$$

for all $u \in H^m(V,E)$.

Proof. Given any $u \in H^m(V,E)$, we define Mu from the equality (4.1), namely

$$Mu = u - Hu - TAu.$$

Note that H is a smoothing operator in the sense that it extends naturally to a continuous mapping

$$H^{-\infty}(V,E) \to \mathring{H}^{\infty}(V,E),$$

where $\mathring{H}^{\infty}(V,E)$ is the projective limit of the family $\mathring{H}^{s}(V,E)$, $s \in \mathbb{Z}_+$, and $H^{-\infty}(V,E)$ the dual space under the pairing induced from $L^2(V,E)$. Hence it follows, by Lemma 3.4, that M is a well-defined continuous mapping of $H^m(V,E)$.

We shall have established the theorem if we prove that the operator M defined from (4.1) is actually an appropriate extension of the operator M

given before Theorem 4.1. This is an easy consequence of Stokes' formula. Indeed, pick a $u \in H^m(V,E)$ vanishing near $\partial\sigma$. Combining Stokes' formula and Theorem 3.3, we get

$$\begin{aligned}(u - Hu - TAu, v)_{L^2(V,E)} &= (u, v - Hv)_{L^2(V,E)} - (Au, AGv)_{L^2(V,F)} \\ &= (u, v - Hv - \Delta Gv)_{L^2(V,E)} - \int_{\partial V} G_A(*_F(AGv), u) \\ &= \left(-T\left([\partial V]^A u\right), v\right)_{L^2(V,E)}\end{aligned}$$

for all $v \in C^\infty_{\mathrm{comp}}(V,E)$. This shows that $Mu = -T\left([\partial V]^A u\right)$ in (the interior of) V, as desired.

□

We now consider the inhomogeneous Dirichlet problem for the Laplacian Δ on V.

To this end, we first give a rigorous meaning to the boundary condition $t(u) = u_0$ on ∂V. If $\partial\sigma$ is sufficiently smooth, t induces a topological isomorphism

$$\frac{H^m(V,E)}{\overset{\circ}{H}{}^m(V,E)} \overset{\cong}{\to} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial V, F_j),$$

which is due to (2.1). Hence we can more generally interpret t as the quotient mapping

$$t: \ H^m(V,E) \to \frac{H^m(V,E)}{\overset{\circ}{H}{}^m(V,E)}, \tag{4.2}$$

the quotient on the right substituting the space of Dirichlet data on ∂V. We make use of the Hilbert structure in $H^m(V,E)$ to construct a continuous right inverse t^{-1} for t.

Problem 4.2 *Given $F \in H^{-m}(V,E)$ and $u_0 \in H^m(V,E)/\overset{\circ}{H}{}^m(V,E)$, find a section $u \in H^m(V,E)$ such that*

$$\left\{\begin{array}{rcll} \Delta u &=& F & in \ \ V, \\ t(u) &=& u_0 & on \ \ \partial V. \end{array}\right.$$

Lemma 4.3 *Problem 4.2 is solvable if and only if $F \in \mathcal{H}^\perp(V)$. Moreover, for each $F \in \mathcal{H}^\perp(V)$,*

$$u = GF + M\left(t^{-1}u_0\right)$$

is the solution to Problem 4.2 belonging to $H^m(V,E) \cap \mathcal{H}^\perp(V)$ and thus satisfying

$$\|u\|_{H^m(V,E)} \le c\left(\|F\|_{H^{-m}(V,E)} + \|u_0\|_{\frac{H^m(V,E)}{\overset{\circ}{H}{}^m(V,E)}}\right). \tag{4.3}$$

Proof. The necessity of the condition $F \in \mathcal{H}^{\perp}(V)$ is obvious. What is left is to show that under this condition $u = GF + M\,(t^{-1}u_0)$ is the solution to Problem 4.2 in $H^m(V, E) \cap \mathcal{H}^{\perp}(V)$.

Indeed, Theorem 3.3 shows that $GF \in \overset{\circ}{H}{}^m(V, E)$ is "orthogonal" to $\mathcal{H}(M)$ and satisfies $\Delta(GF) = F$.

On the other hand, given any Dirichlet data u_0, we find a $U \in H^m(V, E)$ such that $t(U) = u_0$ on ∂V. Note that MU is actually independent of the particular choice of U, for if $U', U'' \in H^m(V, E)$ satisfy

$$\begin{array}{rcl} t(U') & = & u_0, \\ t(U'') & = & u_0 \end{array}$$

then $U' - U'' \in \overset{\circ}{H}{}^m(V, E)$ whence

$$\begin{array}{rcl} MU' & = & MU'' + M\,(U' - U'') \\ & = & MU'' + (U' - U'') - H\,(U' - U'') - G\Delta\,(U' - U'') \\ & = & MU'', \end{array}$$

the last equality being a consequence of Theorem 3.3. Using Theorem 4.1 we get

$$\begin{array}{rcl} \Delta\, MU & = & \Delta\,(U - HU - G\Delta U) \\ & = & \Delta U - (\Delta U - H\,\Delta U) \\ & = & 0 \end{array}$$

and

$$\begin{array}{rcl} t\,(MU) & = & t\,(U - HU - G\Delta U) \\ & = & t\,(U) \\ & = & u_0. \end{array}$$

Finally, the section MU is "orthogonal" to $\mathcal{H}(V)$ because so are both $U - HU$ and $G\Delta U$.

Summarising we conclude that $u = GF + MU$ gives a canonical solution to Problem 4.2, as desired. The estimate (4.3) is a consequence of the Open Map Theorem.

$\square$

Let D be a relatively compact domain (i.e. open connected subset) in $\overset{\circ}{X}$ with a smooth boundary $(S =)\ \partial D$ containing σ.

For

$$\begin{array}{rcl} u & \in & H^m(D, E), \\ f & \in & L^2(D, F), \end{array}$$

we consider the integrals

$$\begin{array}{rcl} H_D u & = & H\left(\chi_D u\right), \\ T_D f & = & T\left(\chi_D f\right), \\ M_D u & = & -T\left([\partial D]^A u\right) \end{array}$$

in V, where χ_D is the characteristic function of D in X. Analysis similar to that in the proof of Theorem 4.1 actually shows that

$$\chi_D u = H_D u + T_D A u + M_D u \tag{4.4}$$

over V, for every $u \in H^m(D,E)$.

Lemma 4.4 *As defined above, the integrals H_D, T_D and M_D induce bounded operators*

$$\begin{array}{rcrcl} H_D & : & H^m(D,E) & \to & H^m((D,\sigma),E), \\ T_D & : & L^2(D,F) & \to & H^m((D,\sigma),E), \\ M_D & : & H^m(D,E) & \to & H^m(D,E). \end{array}$$

Proof. We first observe that the space $H^m((D,\sigma),E)$ coincides with the restriction of $\overset{\circ}{H}{}^m(V,E)$ to D.

Since H extends to a continuous mapping $H^{-\infty}(V,E) \to \mathcal{H}(V)$, the boundedness of H_D is clear.

Suppose $f \in L^2(D,F)$. Then $\chi_D f$ is naturally regarded as the extension of f to a section in $L^2(X,F)$ by zero. We have $T_D f = T\left(\chi_D f\right)$, and so the mapping $T_D : L^2(D,F) \to H^m((D,\sigma),E)$ is continuous, which is due to Lemma 3.4.

Finally, in order to complete the proof it is sufficient to invoke the equality $M_D = \mathrm{Id} - H_D - T_D A$ in D.

□

5 Examples

Example 5.1 Let X be a bounded domain with smooth boundary in $\mathbb{R}^n$, $n > 1$, and A an operator with injective symbol in a neighbourhood $\tilde{X}$ of $\bar{X}$. Assume that A fulfills the uniqueness condition for the Cauchy problem in the small on $\tilde{X}$. Write G for the Green function of the Dirichlet problem for the Laplacian $\Delta = A^*A$ in X. In [3], a scalar product $h_D(\cdot,\cdot)$ on $H^m(D,E)$ is constructed, such that the corresponding norm is equivalent to the original one and the operator T_D is adjoint to $A\colon H^m(D,E) \to L^2(D,F)$ with respect to $h_D(\cdot,\cdot)$, i.e.

$$h_D(T_D f, v) = \int_D (f, Av)_x \, dx$$

for all $f \in L^2(D,F)$ and $v \in H^m(D,F)$. This implies that the iterations of the double layer potential M_D in $H^m(D,F)$ converge to the projection onto the subspace $H^m(D,E) \cap \mathcal{S}_A(D)$. This case corresponds to the Hodge decomposition for the Dirichlet problem in X with empty crack σ and $\mathcal{H}(V)$ being trivial.

$\square$

Recall that $H^m((D,\sigma),E)$ just amounts to the subspace of $H^m(D,E)$ consisting of all u with $t(u)=0$ on σ.

Example 5.2 Under the assumptions of Example 5.1, let moreover X have a crack along a closed piece σ of a smooth hypersurface in X. We denote G the Green function of the Dirichlet problem for the Laplacian Δ in $V = X \setminus \sigma$. In our paper [8], a scalar product $h_D(\cdot,\cdot)$ on $H^m((D,\sigma),E)$ is constructed, defining an equivalent topology on this space and such that the operator T_D is actually adjoint to $A:\ H^m((D,\sigma),E) \to L^2(D,F)$ with respect to $h_D(\cdot,\cdot)$, i.e.,

$$h_D(T_D f, v) = \int_D (f, Av)_x \, dx$$

for all $f \in L^2(D,F)$ and $v \in H^m((D,\sigma),E)$. When combined with a general result of functional analysis, this implies that the limit of iterations M_D^N in the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$ is equal to zero. This case corresponds to the Hodge decomposition for the Dirichlet problem in X with a crack along σ and $H=0$.

$\square$

In the next section we will prove similar results for the integrals T_D and M_D in our more general setting.

6 Construction of the scalar product $h_D(\cdot,\cdot)$

We first apply Lemma 4.3 to $X \setminus D$, a C^∞ manifold with boundary. Namely, write $\mathcal{S}^m_\Delta(\hat{X} \setminus D)$ for the subspace of $H^m(X \setminus D, E)$ consisting of all u, such that $\Delta u = 0$ in the interior of $X \setminus D$ and $t(u) = 0$ on ∂X. By Lemma 4.3, we get a topological isomorphism

$$\mathcal{S}^m_\Delta(\hat{X} \setminus D) \cap \mathcal{H}^\perp(X \setminus D) \xrightarrow{t_+} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D, F_j)$$

given by $u \mapsto t(u)\,|_{\partial D}$. Finally, composing the inverse t_+^{-1} with the trace operator

$$H^m(D,E) \xrightarrow{t_-} \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D, F_j)$$

we arrive at a continuous linear mapping

$$H^m(D,E) \ni u \mapsto \mathcal{E}(u) \in \mathcal{S}^m_\Delta(\hat{X}\setminus D) \cap \mathcal{H}^\perp(X\setminus D). \tag{6.1}$$

For $u \in H^m(D,E)$, we now set

$$e(u)(x) = \begin{cases} u(x) & \text{if } x \in D, \\ \mathcal{E}(u)(x) & \text{if } x \in X\setminus \bar{D}. \end{cases}$$

Since $t(\mathcal{E}(u)) = t(u)$ on ∂D, it follows that $e(u) \in H^m(X,E)$. Furthermore, we have

$$e(u) \in \mathring{H}^m(V,E)$$

for all $u \in H^m((D,\sigma),E)$.

Theorem 6.1 *The Hermitian form $h_D(u,v) = h(e(u),e(v))$ is a scalar product in $H^m((D,\sigma),E)$ defining a topology equivalent to the original one.*

Proof. Theorem 3.5 implies the existence of a positive constant c with the property that

$$\begin{aligned} \|u\|^2_{H^m(D,E)} &\leq \|e(u)\|^2_{H^m(V,E)} \\ &\leq c\, h(e(u),e(u)) \end{aligned}$$

for all $u \in H^m((D,\sigma),E)$.

On the other hand,

$$\begin{aligned} h_D(u,u) &\leq C\, \|e(u)\|^2_{H^m(V,E)} \\ &\leq 2C\left(\|u\|^2_{H^m(D,E)} + \|\mathcal{E}(u)\|^2_{H^m(X\setminus D,E)}\right) \end{aligned}$$

for all $u \in H^m((D,\sigma),E)$, with C a constant independent of u. Using Lemma 4.3 and the continuity of the trace operator we see that

$$\begin{aligned} \|\mathcal{E}(u)\|_{H^m(X\setminus D,E)} &\leq c \sum_{j=0}^{m-1} \|B_j u\|_{H^{m-m_j-1/2}(\partial D,F_j)} \\ &\leq C\, \|u\|_{H^m(D,E)} \end{aligned}$$

for all $u \in H^m(D,E)$, the constants c and C need not be the same in different applications. This finishes the proof.

$\square$

Theorem 6.2 *Assume that $u \in H^m(D,E)$ and $f \in L^2(D,F)$. For every $v \in H^m((D,\sigma),E)$, it follows that*

$$\begin{aligned} h_D(T_D f, v) &= \int_D (f, Av)_x\, dx, \\ h_D((H_D + M_D)u, v) &= \int_{X\setminus D} (A\mathcal{E}(u), A\mathcal{E}(v))_x\, dx + \int_X (He(u), He(v))_x\, dx. \end{aligned}$$

Proof. Suppose $f \in C^\infty_{\mathrm{comp}}(D,F)$. Then $T_D f \in \overset{\circ}{H}{}^m(V,E)$. Let us show that

$$e\,(Tf\,|_D) = Tf.$$

For this purpose, it is sufficient to check that the restriction of Tf to $X \setminus \bar{D}$ lies in $\mathcal{S}^m_\Delta(\hat{X} \setminus D) \cap \mathcal{H}^\perp(X \setminus D)$.

However, $Tf = G\,(A^* f)$ and therefore, as we have already seen in the proof of Theorem 3.5,

$$\begin{aligned} \Delta\, Tf &= A^* f - HA^* f \\ &= A^* f \end{aligned}$$

on X. Since f has a compact support in D we readily deduce that $\Delta\, Tf = 0$ in $X \setminus \bar{D}$.

Note that $\mathcal{H}(X \setminus D) \subset \mathcal{H}(V)$. Indeed, every element $u \in \mathcal{H}(X \setminus D)$ can be extended by zero from $X \setminus D$ to all of X as a solution to $Au = 0$ on X. Since $G\,(A^* f)$ is "orthogonal" to $\mathcal{H}(V)$ we conclude that $Tf\,|_{X\setminus\bar{D}} \in \mathcal{H}^\perp(X \setminus D)$, as desired.

Further, if $v \in H^m((D,\sigma),E)$ then $e(v) \in \overset{\circ}{H}{}^m(V,E)$ and Theorem 3.5 implies

$$\begin{aligned} h_D\,(T_D f, v) &= h\,(Tf, e(v)) \\ &= \int_X (f, Ae(v))_x \, dx \\ &= \int_D (f, Av)_x \, dx. \end{aligned}$$

Since $C^\infty_{\mathrm{comp}}(D,F)$ is dense in $L^2(D,F)$ and the operator T_D is bounded, this formula actually holds for all $f \in L^2(D,F)$.

Finally, (4.4) implies that

$$\begin{aligned} h_D\,((H_D + M_D)u, v) &= h_D\,(u - T_D A u, v) \\ &= \int_{X\setminus D} (A\mathcal{E}(u), A\mathcal{E}(v))_x \, dx + \int_X (He(u), He(v))_x \, dx \end{aligned}$$

for all $v \in H^m((D,\sigma),E)$, as desired.

□

7 Iterations of potentials

Corollary 7.1 *The operators*

$$\begin{aligned} T_D A &: H^m((D,\sigma),E) \to H^m((D,\sigma),E), \\ H_D + M_D &: H^m((D,\sigma),E) \to H^m((D,\sigma),E) \end{aligned}$$

are selfadjoint and non-negative with respect to $h_D(\cdot,\cdot)$*, and the norms of* $T_D A$ *and* $H_D + M_D$ *do not exceed* 1.

Proof. This follows immediately from Theorems 6.1 and 6.2. □

Similarly to $\mathcal{H}^\perp(V)$, we denote $\mathcal{H}^\perp(D)$ the space of all $F \in H^{-m}(D,E)$ such that

$$\int_D (F, v)_x \, dx = 0$$

for any $v \in \mathcal{H}(D)$. It is easy to see that $H^m((D,\sigma),E) \cap \mathcal{H}^\perp(D)$ just amounts to the orthogonal complement of $\mathcal{H}(D)$ in $H^m((D,\sigma),E)$ with respect to the scalar product $h_D(\cdot,\cdot)$ thereon. Indeed, we have $\mathcal{H}(D) \hookrightarrow \mathcal{H}(V)$ because every element $v \in \mathcal{H}(D)$ may be extended by zero from D to X as a solution to $Av = 0$ on X. It follows that $\mathcal{E}(v) = 0$ for all $v \in \mathcal{H}(D)$, whence $Ae(v) = 0$ on X and

$$\begin{aligned} h_D(u,v) &= \int_X (He(u), He(v))_x \, dx \\ &= \int_D (u,v)_x \, dx, \end{aligned}$$

as desired.

Lemma 4.4 allows one to consider iterations of $T_D A$ and $H_D + M_D$ in the space $H^m((D,\sigma),E)$. Given a closed subspace Σ of $H^m((D,\sigma),E)$, we write π_Σ for the orthogonal projection of $H^m((D,\sigma),E)$ onto Σ with respect to the scalar product $h_D(\cdot,\cdot)$.

Corollary 7.2 *In the strong operator topology in* $\mathcal{L}(H^m((D,\sigma),E))$*, we have*

$$\begin{aligned} \lim_{N\to\infty} (T_D A)^N &= \pi_{\ker(H_D+M_D)}, \\ \lim_{N\to\infty} (H_D + M_D)^N &= \pi_{H^m((D,\sigma),E)\cap\mathcal{S}_A(D)}. \end{aligned}$$

Moreover, in the strong operator topology of $\mathcal{L}(L^2(D,F))$*,*

$$\lim_{N\to\infty} (\mathrm{Id} - AT_D)^N = \pi_{\ker T_D}.$$

Proof. It follows from Corollary 7.1 that

$$\begin{aligned} \lim_{N\to\infty} (T_D A)^N &= \pi_{\ker(\mathrm{Id}-T_D A)}, \\ \lim_{N\to\infty} (H_D + M_D)^N &= \pi_{\ker(\mathrm{Id}-H_D-M_D)}, \\ \lim_{N\to\infty} (\mathrm{Id} - AT_D)^N &= \pi_{\ker AT_D} \end{aligned}$$

in the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$ or $\mathcal{L}(L^2(D,F))$, respectively (see, for instance, §2 of [3] or [4] for compact operators). Theorem 6.2 and (4.4) imply

$$\begin{array}{rcl}\ker(\mathrm{Id} - T_D A) & = & \ker(H_D + M_D), \\ \ker T_D A & = & H^m((D,\sigma),E) \cap \mathcal{S}_A(D), \\ \ker A T_D & = & \ker T_D,\end{array}$$

showing the corollary.

□

Obliviously, if the coefficients of A are real analytic and σ has at least one interior point on ∂D then $H^m((D,\sigma),E) \cap \mathcal{S}_A(D) = \{0\}$. If moreover every connected component of $X \setminus \bar{D}$ meets the boundary of V, i.e., $\partial X \cup \sigma$, then $H_D = 0$ and

$$\ker M_D = \overset{\circ}{H}{}^m(D,E).$$

Indeed, according to Theorem 6.2, if $u \in H^m((D,\sigma),E)$ and $M_D u = 0$ then $A\mathcal{E}(u) = 0$ in $X \setminus \bar{D}$ and $t(\mathcal{E}(u)) = 0$ on $\partial X \cup \sigma$. Hence it follows that $\mathcal{E}(u) \equiv 0$ in $X \setminus \bar{D}$, and so $t(\mathcal{E}(u)) = 0$ on ∂D. From this we conclude that $t(u) = 0$ on ∂D whence $u \in \overset{\circ}{H}{}^m(D,E)$. Conversely, if $u \in \overset{\circ}{H}{}^m(D,E)$ then $M_D u = 0$, as desired.

Theorem 7.3 *In the strong operator topology of $\mathcal{L}(H^m((D,\sigma),E))$, we have*

$$\mathrm{Id} = H_D + \pi_{\ker(H_D+M_D)} + \sum_{\nu=0}^{\infty} (T_D A)^\nu M_D, \tag{7.1}$$

$$\mathrm{Id} = \pi_{H^m((D,\sigma),E)\cap\mathcal{S}_A(D)} + \sum_{\nu=0}^{\infty} (H_D + M_D)^\nu T_D A. \tag{7.2}$$

Moreover, in the strong operator topology of $\mathcal{L}(L^2(D,F))$,

$$\mathrm{Id} = \pi_{\ker T_D} + \sum_{\nu=0}^{\infty} A\,(H_D + M_D)^\nu T_D. \tag{7.3}$$

Proof. Write

$$\mathrm{Id} = (\mathrm{Id} - A T_D)^N + \sum_{\nu=0}^{N-1} (\mathrm{Id} - A T_D)^\nu A T_D, \tag{7.4}$$

for every $N = 1, 2, \ldots$. It is easily seen from (4.4) that

$$\begin{array}{rcl}(\mathrm{Id} - A T_D)^\nu A T_D & = & A\,(\mathrm{Id} - T_D A)^\nu T_D \\ & = & A\,(H_D + M_D)^\nu T_D.\end{array}$$

Using Corollary 7.2 we can pass to the limit in (7.4), when $N \to \infty$, thus obtaining (7.3). The proofs for (7.1) and (7.2) are similar.

□

8 Cauchy problem

We first introduce the space of Cauchy data on σ, for our differential operator A. Since A is given the domain $H^m(D,E)$, the space of zero Cauchy data is $H^m((D,\sigma),E)$. Recall that $H^m((D,\sigma),E)$ is proved to be the restriction of $\mathring{H}^m(V,E)$ to D.

Similarly to (4.2) we define the space of Cauchy data on σ as the quotient space

$$\frac{H^m(D,E)}{H^m((D,\sigma),E)},$$

t being thought of as the quotient mapping

$$t: \ H^m(D,E) \to \frac{H^m(D,E)}{H^m((D,\sigma),E)}. \tag{8.1}$$

Once again we use the Hilbert structure in $H^m(D,E)$ to construct a continuous right inverse t^{-1} for t.

If the boundary of σ on ∂D is sufficiently smooth then the quotient space in (8.1) can be identified with $\oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\sigma,F_j)$.

Consider the following Cauchy problem, for the operator A and the Dirichlet system $(B_j)_{j=0,\ldots,m-1}$.

Problem 8.1 *Given $f \in L^2(D,F)$ and $u_0 \in H^m(D,E)/H^m((D,\sigma),E)$, find $u \in H^m(D,E)$ satisfying*

$$\left\{ \begin{array}{rcll} Pu & = & f & \textit{in} \ \ D, \\ t(u) & = & u_0 & \textit{on} \ \ \sigma. \end{array} \right.$$

This problem is ill-posed if σ is different from the whole boundary. Using Theorem 7.3 we obtain approximate solutions to the problem. To this end, we observe that

$$\frac{H^m(V,E)}{\mathring{H}^m(V,E)} \hookrightarrow \frac{H^m(D,E)}{H^m((D,\sigma),E)}$$

is a well-defined mapping "onto", which substitutes restriction of sections over ∂V to σ. Pick a $U \in H^m(V,E)$ such that $t(U) = u_0$ on σ. Lemma 4.3 yields that $MU \in H^m(V,E)$ satisfies $\Delta MU = 0$ in V and $t(MU) = u_0$ on σ, the last property being sufficient. Problem 8.1 thus reduces to that with zero boundary conditions.

Problem 8.2 *Given any $f \in L^2(D,F)$, find a section $u \in H^m((D,\sigma),E)$ such that $Au = f$ in D.*

Note that for the problem to be solvable it is necessary that $f \perp \ker T_D$. Indeed,

$$\begin{aligned} \int_D (f, g)_x \, dx &= \int_D (Au, g)_x \, dx \\ &= h_D(u, T_D g) \\ &= 0 \end{aligned}$$

for all $g \in L^2(D, F)$ satisfying $T_D g = 0$, the second equality being due to Theorem 6.2.

Theorem 8.3 *Suppose $f \in L^2(D, F)$. Problem 8.2 is solvable if and only if $f \perp \ker T_D$ and the series*

$$Rf = \sum_{\nu=0}^{\infty} (H_D + M_D)^{\nu} \, T_D f$$

converges in $H^m((D, \sigma), E)$. Moreover, if these conditions hold then Rf is a solution to Problem 8.2.

Proof. As mentioned above, the necessity follows from Theorems 6.1 and 7.3.

Conversely, let both conditions of the theorem be fulfilled. Then (7.3) implies

$$f = \sum_{\nu=0}^{\infty} A \, (H_D + M_D)^{\nu} \, T_D f.$$

Since the series Rf converges in $H^m((D, \sigma), E)$ we conclude that $f = ARf$, as desired.

□

In the case considered in Example 5.2 a similar result has been proved in [8].

Corollary 7.2 shows that the solution $u = Rf$ lies in the orthogonal complement of the subspace $H^m((D, \sigma), E) \cap \mathcal{S}_A(D)$ in $H^m((D, \sigma), E)$ with respect to the scalar product $h_D(\cdot, \cdot)$. Clearly, Problem 8.2 possesses at most one solution belonging to this orthogonal complement. The partial sums $R_N f$ of the series Rf may be regarded as approximate solutions to Problem 8.2, provided that $f \perp \ker T_D$. In fact, it follows easily from Corollary 7.2 and Theorem 7.3 that $R_N f$ belongs to the orthogonal complement in question, for each $N = 0, 1, \ldots$, and

$$\lim_{N \to \infty} \| f - (\pi_{\ker T_D} f + AR_N f) \, \|_{L^2(D,F)} = 0$$

for all $f \in L^2(D,F)$. Indeed,

$$\| f - (\pi_{\ker T_D} f + A R_N f) \|_{L^2(D,F)} = \| \sum_{\nu=N+1}^{\infty} A\, M_D^{\nu}\, T_D f \|_{L^2(D,F)},$$

and the last expression is the rest of a converging series.

If A is included into an elliptic complex

$$C^{\infty}_{\mathrm{loc}}(X,E) \xrightarrow{A} C^{\infty}_{\mathrm{loc}}(X,F) \xrightarrow{B} C^{\infty}_{\mathrm{loc}}(X,G)$$

then the condition $f \perp \ker T_D$ in Theorem 8.3 may be replaced by

1) $Bf = 0$ in D;

2) $f \perp \ker T_D \cap \mathcal{S}_B(D)$.

Write

$$n(g) = \oplus_{j=0}^{m-1} *_{F_j}^{-1} C_j *_F (g)$$

for the formal adjoint of t with respect to the Green formula for A in D, cf. [9, 9.2.3]. Set

$$\mathcal{H}^1(D,\sigma) = \{ g \in L^2(D,F) : \ A^* g = 0, \ Bg = 0, \ \text{and } n(g) = 0 \text{ on } \partial D \setminus \sigma \}.$$

We call $\mathcal{H}^1(D,\sigma)$ the *harmonic space* in the Cauchy problem with data on σ. By the ellipticity assumption, the elements of $\mathcal{H}^1(D,\sigma)$ are of class C^∞ in D.

Lemma 8.4 $\ker T_D \cap \mathcal{S}_B(D) = \mathcal{H}^1(D,\sigma)$.

Proof. Let $g \in \ker T_D \cap \mathcal{S}_B(D)$. From Theorem 6.2 it follows that $A^* g = 0$ in the sense of distributions on D. By the ellipticity of $B \oplus A^*$ we conclude that $g \in C^{\infty}_{\mathrm{loc}}(D,F)$.

We next claim that $n(g) = 0$ weakly on $\partial D \setminus \sigma$. To prove this, we denote by D_ε the set of all $x \in D$ such that $\mathrm{dist}(x, \partial D) > \varepsilon$. For $\varepsilon > 0$ small enough, D_ε is also a domain with C^∞ boundary. We shall have established the equality $n(g) = 0$ on $\partial D \setminus \sigma$ if we show that

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (t(u), n(g))_x \, ds_\varepsilon = 0$$

for all $u \in C^\infty(\bar{D}, E)$ vanishing near σ. Here, ds_ε is the area element of the surface ∂D_ε.

Since g is C^∞ in D, we get

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} (t(u), n(g))_x \, ds_\varepsilon &= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (Au, g)_x \, dx \\
&= \int_D (Au, g)_x \, dx \\
&= h_D(u, T_D g) \\
&= 0,
\end{aligned}$$

the first equality being due to Stokes' formula and the equality $A^* g = 0$, the second equality being a consequence of the fact that $g \in L^2(D,F)$, and the third one being due to Theorem 6.2. We have thus proved that $\ker T_D \cap \mathcal{S}_B(D)$ is a subset of $\mathcal{H}^1(D,\sigma)$.

Let us prove the opposite inclusion. Pick a $g \in \mathcal{H}^1(D,\sigma)$. By ellipticity we conclude that $g \in C^\infty_{\mathrm{loc}}(D,F)$. For every $u \in C^\infty_{\mathrm{loc}}(\bar{D},E)$ vanishing near σ, we have

$$
\begin{aligned}
h_D\left(u, T_D g\right) &= \lim_{\varepsilon \to 0} \int_{D_\varepsilon} \left(Au, g\right)_x \, dx \\
&= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \left(t(u), n(g)\right)_x \, ds_\varepsilon \\
&= 0.
\end{aligned}
$$

Since such sections u are dense in $H^m((D,\sigma),E)$, it follows that $T_D g = 0$, as desired.

□

It is worth pointing out that the space $\mathcal{H}^1(D,\sigma)$ fails to be finite-dimensional in general.

9 Applications to Zaremba problem

In this section we assume that σ is the closure of an open subset in ∂D with smooth boundary.

Let $H^{-m}((D, \partial D \setminus \sigma), E)$ be the dual space for $H^m((D,\sigma),E)$ with respect to the pairing in $L^2(D,E)$. It coincides with the completion of $C^\infty_{\mathrm{comp}}(D \cup \sigma, E)$ with respect to the norm

$$
\|F\|_{H^{-m}((D,\partial D\setminus\sigma),E)} = \sup_{v \in C^\infty_{\mathrm{comp}}(\bar{D}\setminus\sigma,E)} \frac{\left|\int_D (F,v)_x \, dx\right|}{\|v\|_{H^m((D,\sigma),E)}}.
$$

Recall that for $s \geq 0$ we write $H^{-s}(\partial D \setminus \sigma, F_j)$ for the dual of $\overset{\circ}{H}{}^s(\partial D \setminus \sigma, F_j)$ with respect to the pairing in $L^2(\partial D \setminus \sigma, F_j)$, cf. Section 3. One can prove that

$$
H^{-s}(\partial D \setminus \sigma, F_j) \overset{\text{top.}}{\cong} \frac{H^{-s}(\partial D, F_j)}{H^{-s}_\sigma(\partial D, F_j)}
$$

where $H^{-s}_\sigma(\partial D, F_j)$ is the subspace of $H^{-s}(\partial D, F_j)$ consisting of the elements with a support in σ.

By the above, the sesquilinear form

$$
\int_{\partial D} \left(u_1, t(v)\right)_x \, ds
$$

is well defined for all

$$\begin{array}{rcl} u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \\ v & \in & H^m((D,\sigma),E). \end{array}$$

We are now in a position to consider the following generalised Zaremba problem in D.

Problem 9.1 *Given*

$$\begin{array}{rcl} F & \in & H^{-m}((D,\partial D \setminus \sigma),E), \\ u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \end{array}$$

find $u \in H^m(D,E)$ such that

$$\left\{ \begin{array}{rclcl} \Delta u & = & F & \mathit{in} & D, \\ t(u) & = & 0 & \mathit{on} & \sigma, \\ n(Au) & = & u_1 & \mathit{on} & \partial D \setminus \sigma. \end{array} \right.$$

The equation $\Delta u = F$ has to be understood in the sense of distributions in D, while the boundary conditions are interpreted in the following weak sense: Find $u \in H^m((D,\sigma),E)$ satisfying

$$\int_D (Au, Av)_x \, dx = \int_D (F, v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds \tag{9.1}$$

for all $v \in H^m((D,\sigma),E)$.

We emphasise that the trace of $n(Au)$ on $\partial D \setminus \sigma$ is not defined for any $u \in H^m((D,\sigma),E)$, because the order of $n \circ A$ is equal to $2m-1$. To cope with this, a familiar way is to assign an operator L with a dense domain $\operatorname{Dom} L \hookrightarrow H^m((D,\sigma),E)$ to Problem 9.1, such that $n(Au)$ is well defined for all $u \in \operatorname{Dom} L$. In fact, $\operatorname{Dom} L$ is defined to be the completion of $C^\infty_{\mathrm{comp}}(\bar{D} \setminus \sigma, E)$ with respect to the graph norm of $u \mapsto (u, n(Au))$ in $H^m((D,\sigma),E) \oplus \mathfrak{N}$, where

$$\mathfrak{N} = \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j).$$

For more details, see Roitberg [5] and elsewhere. Then, (9.1) defines a continuous operator $\operatorname{Dom} L \to H^{-m}((D,\partial D \setminus \sigma),E) \oplus \mathfrak{N}$ by $Lu = (\Delta u, n(Au))$.

If A is the gradient operator in $\mathbb{R}^n$, then (9.1) is just the classical Zaremba problem in D.

Lemma 9.2 *Suppose $F = 0$ and $u_1 = 0$. Then $u \in H^m(D,E)$ is a solution to Problem 9.1 if and only if $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$.*

Proof. Obviously, any $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ is a solution of Problem 9.1 with $F = 0$ and $u_1 = 0$.

Conversely, let u be a solution to Problem 9.1 with $F = 0$ and $u_1 = 0$. Substituting $v = u$ to (9.1) implies $Au = 0$ whence $u \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$, as desired.

□

Lemma 9.2 shows that Problem 9.1 is not Fredholm in general, for the space $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ need not be finite-dimensional.

For any $v \in \mathring{H}^m(V,E)$, the restriction $v\,|_D$ belongs to $H^m((D,\sigma),E)$. Hence to each $F \in H^{-m}((D,\partial D \setminus \sigma),E)$ we can assign an element $\tilde{F} \in H^{-m}(V,E)$ by

$$(\tilde{F}, v) = (F, v\,|_D)$$

for all $v \in \mathring{H}^m(V,E)$. We will write $\tilde{F}$ simply $\chi_D F$ when no confusion can arise. Therefore, the integral

$$G\,(\chi_D F) = \int_D \left(F, *_E^{-1} K_G(x,\cdot)\right)_y dy$$

defines an element of $H^m((D,\sigma),E)$, for any $F \in H^{-m}((D,\partial D \setminus \sigma),E)$.

Furthermore, since $u \mapsto t(u) \oplus n(Au)$ is a Dirichlet system of order $2m-1$ on ∂D, for every data $u_1 \in \mathfrak{N}$ there exists a $U \in \operatorname{Dom} L$ with the property that $n(AU) = u_1$ on $\partial D \setminus \sigma$ (see for instance Lemma 9.2.17 of [9]). This means that

$$\int_D (AU, Av)_x\, dx = \int_D (\Delta U, v)_x\, dx + \int_{\partial D} (u_1, t(v))_x\, ds$$

for all $v \in H^m((D,\sigma),E)$. Set

$$\begin{array}{rcl} P_{\mathrm{sl}} u_1\,(x) & = & -\displaystyle\int_{\partial D} G_{A^*}(K_G(x,\cdot), AU) \\ & = & \displaystyle\int_{\partial D} \left(n(AU), t\left(*_E^{-1} K_G(x,\cdot)\right)\right)_y ds, \end{array}$$

for $x \in D$.

This integral is well defined and it does not depend on the particular choice of U. Indeed, since $\Delta U \in H^{-m}((D,\partial D \setminus \sigma),E)$ we conclude, by Stokes' formula, that

$$\begin{array}{rcl} P_{\mathrm{sl}} u_1 & = & GA^*\,(\chi_D AU) - G\,(\chi_D \Delta U) \\ & = & T_D AU - G\,(\chi_D \Delta U)\,, \end{array}$$

which is in $H^m((D,\sigma),E)$.

Let now $U \in \operatorname{Dom} L$ be such that $n(AU) = 0$ on $\partial D \setminus \sigma$. Then using Theorem 3.3 yields

$$\begin{aligned}
\int_X \Big(- \int_{\partial D} G_{A^*}(K_G(x,\cdot), AU), v \Big)_x dx &= \int_X (GA^* \chi_D AU - G\chi_D \Delta U, v)_x \, dx \\
&= \int_X (A^* \chi_D AU - \chi_D \Delta U, Gv)_x \, dx \\
&= \int_D (AU, AGv)_x \, dx - \int_D (\Delta U, Gv)_x \, dx \\
&= 0
\end{aligned}$$

for all $v \in C^\infty_{\mathrm{comp}}(X, E)$, because $Gv \in H^m((D,\sigma), E)$. Hence, $P_{\mathrm{sl}} u_1$ is independent of the choice of U.

Theorem 9.3 *Problem 9.1 is solvable if and only if*
1)
$$\int_D (F, v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds = 0$$
for all $v \in H^m((D,\sigma), E) \cap \mathcal{S}_A(D)$*;*
2) *the series*

$$R(F, u_1) = \sum_{\nu=0}^{\infty} (H_D + M_D)^\nu \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right)$$

converges in the $H^m(D, E)$ *-norm.*
If 1) and 2) hold then $R(F, u_1)$ *is a solution to Problem 9.1.*

Proof. Let Problem 9.1 be solvable and let $u \in H^m((D,\sigma), E)$ be a solution. Then

$$\begin{aligned}
T_D A u &= G(\chi_D \Delta u) - \int_{\partial D} G_{A^*}(K_G(x,\cdot), Au) \\
&= G(\chi_D F) + P_{\mathrm{sl}} u_1,
\end{aligned}$$

and so the series $R(F, u_1) = RAu$ converges in the $H^m(D, E)$-norm, which is due to Corollary 7.2.

Conversely, assume that 1) and 2) are fulfilled. Let us prove that the series $R(F, u_1)$ satisfies (9.1). Indeed, by Theorem 6.1

$$\begin{aligned}
\int_D (AR(F, u_1), Av)_x \, dx &= h_D \left(G(\chi_D F) + P_{\mathrm{sl}} u_1, v \right) \\
&= h_D \left(G(\chi_D F) + GA^* (\chi_D AU) - G(\chi_D \Delta U), v \right)
\end{aligned}$$

with a section $U \in \operatorname{Dom} L$ such that $n(AU) = u_1$ on $\partial D \setminus \sigma$.

Using Theorem 3.3 we see that $e(G\tilde{F}) = G\tilde{F}$ for all $\tilde{F} \in H^{-m}(V,E)$ satisfying $\tilde{F} - H\tilde{F} = 0$ in $X \setminus \bar{D}$. We next apply this equality with

$$\tilde{F} = \chi_D F + A^* (\chi_D AU) - \chi_D \Delta U.$$

We have

$$\begin{aligned}
\int_X \left(\tilde{F}, v\right)_x dx &= \int_D (F - \Delta U, v)_x \, dx \\
&= \int_D (F, v)_x \, dx + \int_{\partial D} (n(AU), t(v))_x \, ds \\
&= 0
\end{aligned}$$

for all $v \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$, the last equality being a consequence of condition 1). Hence it follows readily that $H\tilde{F} = 0$ in V, and so $\tilde{F} - H\tilde{F} = 0$ in $X \setminus \bar{D}$.

Therefore, $e(G\tilde{F}) = G\tilde{F}$ and we get

$$\begin{aligned}
\int_D (AR(F,u_1), Av)_x \, dx &= h\left(e(G\tilde{F}), e(v)\right) \\
&= \int_X \left(AG\tilde{F}, Ae(v)\right)_x dx + \int_X \left(HG\tilde{F}, He(v)\right)_x dx \\
&= \int_X \left(\tilde{F}, G\Delta e(v)\right)_x dx \\
&= \int_X \left(\tilde{F}, e(v) - He(v)\right)_x dx \\
&= \int_D (F - \Delta U, e(v))_x \, dx + \int_D (AU, Ae(v))_x \, dx \\
&= \int_D (F, v)_x \, dx + \int_{\partial D} (n(AU), t(v))_x \, ds
\end{aligned}$$

for all $v \in H^m((D,\sigma),E)$. Here, the fifth equality is due to condition 1) and the fact that $He(v) \in H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$, and the last equality is a consequence of Stokes' formula. We have arrived at (9.1), thus proving the theorem.

$\square$

Corollary 7.2 implies that the solution $R(F,u_1)$ to Problem 9.1 lies in the orthogonal complement of $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ in $H^m((D,\sigma),E)$ with respect to the scalar product $h_D(\cdot,\cdot)$. Moreover, $R(F,u_1)$ is the unique solution belonging to this subspace. The partial sums $R_N(F,u_1)$ of the series $R(F,u_1)$ may be regarded as approximate solutions to Problem 9.1, provided that F

and u_1 meet condition 1) of Theorem 9.3. The sequence $R_N(F,u_1)$ has the following property:

$$\begin{aligned} | \int_D (AR_N(F,u_1), Av)_x \, dx - \int_D (F,v)_y \, dy - \int_{\partial D} (u_1, t(v))_x \, ds | \\ \leq c \, \| M_D^{N+1} \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right) \|_{H^m(D,E)} \|v\|_{H^m(D,E)} \end{aligned} \tag{9.2}$$

for all $v \in H^m((D,\sigma),E)$, with c a constant independent of N and v. Indeed, as we have seen above (cf. (7.4)),

$$T_D A R_N(F,u_1) = (G(\chi_D F) + P_{\mathrm{sl}} u_1) - M_D^{N+1} \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right)$$

whence

$$\begin{aligned} & \int_D (AR_N(F,u_1), Av)_x \, dx \\ & \quad = h_D \left(T_D A R_N(F,u_1), v \right) \\ & \quad = h_D \big((G(\chi_D F) + P_{\mathrm{sl}} u_1) - M_D^{N+1} \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right), v \big) \\ & \quad = \int_D (F,v)_x \, dx + \int_{\partial D} (u_1, t(v))_x \, ds - h_D \left(M_D^{N+1} \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right), v \right). \end{aligned}$$

The scalar product $h_D(\cdot,\cdot)$ defines an equivalent norm, hence the estimate (9.2) holds. Since $G(\chi_D F) + P_{\mathrm{sl}} u_1$ is orthogonal to $H^m((D,\sigma),E) \cap \mathcal{S}_A(D)$ we see that

$$\lim_{N\to\infty} \| M_D^{N+1} \left(G(\chi_D F) + P_{\mathrm{sl}} u_1 \right) \|_{H^m(D,E)} = 0.$$

Of course, if Problem 9.1 is Fredholm then the series $R(F,u_1)$ converges for all data F and u_1.

In the setting of Example 5.2 such a theorem was proved in [8]. We can also treat the inhomogeneous Zaremba problem.

Problem 9.4 *Given*

$$\begin{array}{rcl} F & \in & H^{-m}((D, \partial D \setminus \sigma), E), \\ u_0 & \in & \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\sigma, F_j), \\ u_1 & \in & \oplus_{j=0}^{m-1} H^{-m+m_j+1/2}(\partial D \setminus \sigma, F_j), \end{array}$$

find $u \in H^m(D,E)$ *such that*

$$\left\{ \begin{array}{rcll} \Delta u & = & F & \textit{in} \ \ D, \\ t(u) & = & u_0 & \textit{on} \ \ \sigma, \\ n(Au) & = & u_1 & \textit{on} \ \ \partial D \setminus \sigma. \end{array} \right.$$

Indeed, using the potential $M_D t^{-1} u_0$ as in Section 4, we easily reduce it to Problem 9.1.

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